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GUIDANCE SCHEMES FOR ROCKET VEHICLES

by

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TABLE OF CONTENTS

FORWARD

The enclosed material consists of lecture notes prepared for A Short Course
on Inertial Guidance presented by Engineering Extension and Physical Sciences
Extension, University of California, Los Angeles, October 10-21, 1960. The
theory of inertial instruments, platforms, and error analysis are presented
in other parts of the course. This material is concerned mainly with the proper
utilization of inertially derived position and velocity data in a way which will
fulfill the mission objectives.

ROCKET VEHICLE GUIDANCE SCHEMES*

1.0 INTRODUCTION

The guidance scheme or philosophy for a rocket or ballistic vehicle differs from that of a cruise vehicle in two principal respects. The first is a result of the fact that a rocket vehicle thrusts at high levels for relatively small fractions of its total flight time, while essentially the opposite is true for a cruise vehicle. Since significant control forces are available only during powered flight, the rocket vehicle's guidance system must be able to direct the vehicle's course during this time in a way which will precisely influence its position and/or velocity minutes, hours, or even months later.

The second major difference is in the area of dimensions. Rocket guidance systems are three dimensional, with vehicles being guided between two points in inertial space. Cruise vehicle guidance systems, on the other hand, are basically two dimensional in nature, with the third dimension being fixed or externally supplied. Vehicles are guided over a known surface like, for example, the earth.

1.1 Vehicle Characteristics

Rocket vehicles have lengths which range from a few feet in the case of short range missiles to 100 feet or more in the case of ICBM's and space vehicles. Weights again range from a few pounds to hundreds of thousands of pounds. They can have any number of stages, one to four being the most common and may be either liquid or solid propelled. Thrust accelerations during each stage usually vary from 1.5-3 g's at ignition to 8-10 g's at burnout. The structure, particularly for the larger vehicles, is extremely light with at least 90 percent of the gross weight being fuel and oxidizer. The resulting lack of rigidity often places rather severe limitations on the vehicle's maneuvering capability.

Flight path control is usually obtained by pointing the missile in the direction of desired thrust, with changes in attitude being obtained by momentarily deflecting the direction of thrust. This can be accomplished by gimballing the thrust chamber in the case of a liquid or the nozzle in the case of a solid. Alternatively, the flame pattern can be deflected by means of jet vanes or jetavators placed in the thrust stream. Guidance can be radio, inertial or a combination of the two. The discussion in this chapter, while aimed primarily at inertial guidance, is frequently general enough to apply equally well to other types of systems.

1.2 Trajectory Characteristics

The trajectory for a rocket vehicle can be divided into three types of phases as shown in Figure 1: I powered flight, II free flight, and III re-entry.

(a) ICBM

(b) EARTH SATELLITE
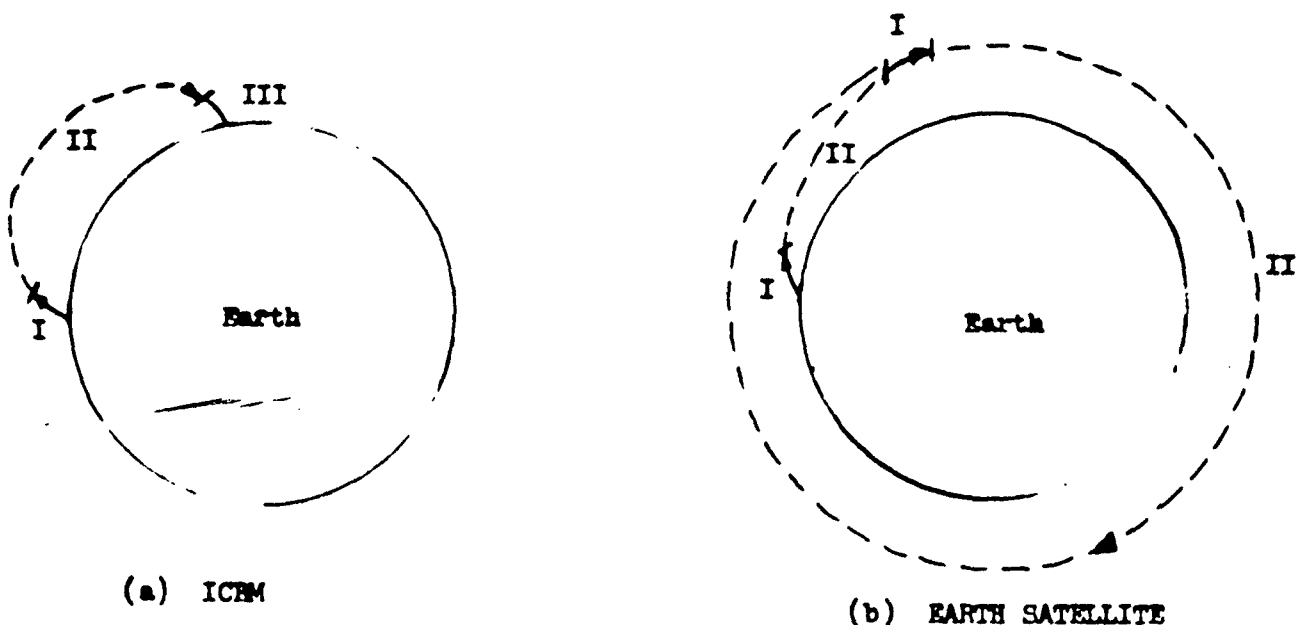
Flight Phases
I - Powered Flight
II - Free or Ballistic Flight
III - Re-entry

Figure 1 - SOME TYPICAL ROCKET TRAJECTORIES

There will be one or more each of the powered and free flight phases. There will be a re-entry phase only if the payload is returning to earth or arriving at some other planet having an atmosphere.

The initial powered phase is the most complex, because of the exit atmosphere. The trajectory usually begins with the missile rising vertically for a few seconds. During this time it rolls to the proper heading. The vehicle then executes its pitch maneuver; after a short transient, usually called transition turn, a gravity or zero lift turn begins and continues until the missile has effectively left the atmosphere. The gravity turn, which is accomplished by causing the missile to thrust always along its velocity vector, minimizes drag effects and aerodynamic heating. The gravity turn is usually continued to some staging point, although this is not always the case, particularly when there is only a single stage. After leaving the atmosphere, structural constraints can be relaxed and a more arbitrary attitude profile can be prescribed. A very high acceleration vehicle, however, can achieve the desired velocity before it ever leaves the atmosphere. This can cause significant steering problems.

When thrust has been terminated, the vehicle begins its free flight, where gravity is the only acting force. The free flight trajectory lies completely within a plane which contains the center of the earth and will be in the shape of a conic - either an ellipse, a parabola, or a hyperbola, depending on whether the velocity is below or above escape velocity, the parabola being the limiting case. In the case of a ballistic missile, the ellipse intersects the earth at the target. Actually the earth's oblateness causes the trajectory to be non-planar and to differ slightly from a true ellipse. Similarly, the influence of other celestial bodies on earth satellites and space probes keeps them from being pure conics. The thrust-coast sequence can be repeated essentially any number of times depending only on the mission.

Any rocket which returns to earth, such as a ballistic missile or a manned
space vehicle, must finally undergo a re-entry phase. Here non-nominal re-
entry conditions such as winds or density variations can also contribute to
system inaccuracy, since these effects are usually not predictable during the
boost guidance period. If re-entry guidance is used, then these effects are
eliminated.

1.3 The Guidance Problem

The thrust of a rocket engine is a complex function of the engine (and propellant)
parameters, the air pressure, the temperature, the vehicle acceleration, and to a
lesser extent, a variety of other quantities. For a given set of engine and
vehicle parameters, any desired trajectory can be synthesized by using simulation
techniques on a high speed digital computer, providing, of course, that the per-
formance limitations of the vehicle are not exceeded and provided the relative
positions of vehicle and target are known. On the computer the desired trajectory
is achieved by time programming vehicle attitude and terminating thrust at the
appropriate time or times. How then is the powered flight of an actual vehicle
controlled so that it too accomplishes its desired mission?

A Simple Example - A simple example for purposes of illustration the German
V-2 missile, developed near the end of World War II[*]. This missile had a take-off
weight of 28,600 pounds, a thrust of 59,800 pounds, and a maximum thrust accelera-
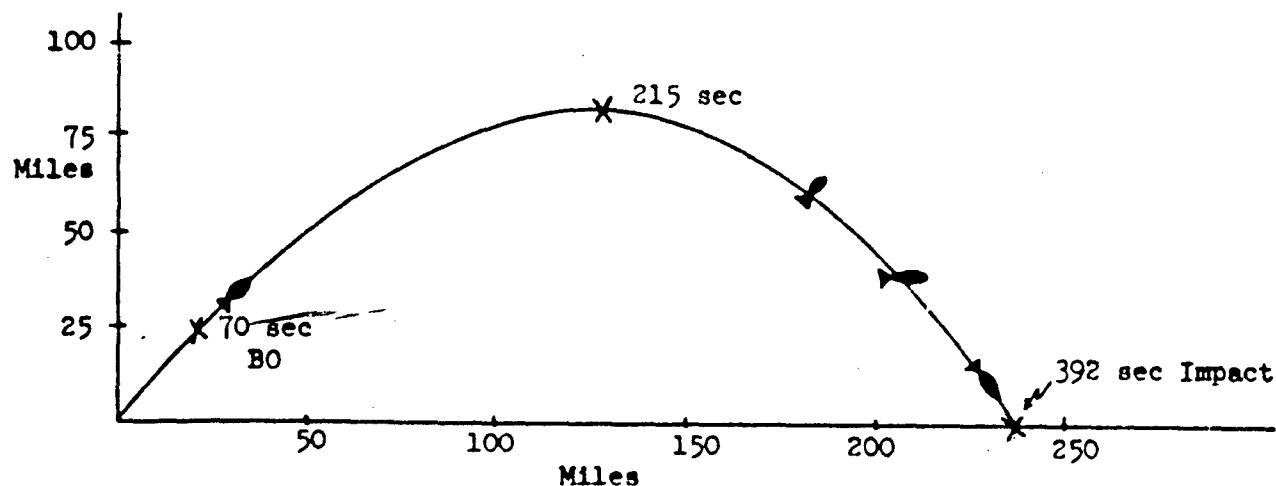tion of 6.4 g's. A typical trajectory is shown in Figure 2.

Figure 2 - V-2 TRAJECTORY

*See for example, "Inertial Guidance for Rocket-Propelled Missiles", by
W. T. Russell, Jet Propulsion, January 1958.

For this case, the missile had a range of 230 miles and nominally burned out at 70 seconds, about 15 miles downrange with a speed (V) of about 6000 fps and a burnout angle (γ) measured from the horizontal of about 54°.

A simple scheme for guiding such a vehicle is again to program attitude and burnout as functions of time as is done in the simulation. A pitch attitude programmer is installed in the missile and its output compared with the gimbal angles of an inertial platform, the difference being used as the control system error signal. Yaw and roll can also be programmed, probably to zero. A clock is used to shut off the engine at 70 seconds or, alternatively, only enough fuel is placed in the missile so that the engine burns out at 70 seconds.

The scheme just outlined performs very well as long as both the missile and its environment are nominal. The impact accuracy is limited only by the performance of the programmer and the attitude reference. A nominal missile is, however, only the average of an ensemble, with any given vehicle differing from the nominal to some degree. Some of the more significant perturbations as far as the trajectory is concerned are given in Table 1.

Table 1
Significant Perturbations
Thrust Variation
Initial Mass Variation
Mass Flow Rate Variation
Thrust Misalignments
Drag Variations
Wind (gusts and shear)

Thrust and mass variations can be in the neighborhood of a few percent, misalignments around a degree, and winds sometimes in the hundreds of mph. While it is clear that such perturbations can cause impact errors, it is necessary to examine the equations of motion if a more quantitative indication of accuracy is desired.

An Error Analysis - For a range of 230 miles, the earth can be assumed flat to a first approximation. If x is downrange and z is up and if the re-entry atmosphere is ignored, the equations which describe the missile's flight from burnout (BO) to impact are

$$x = x_o + \dot{x}_o t_{ff} \tag{1}$$

$$z = z_o + \dot{z}_o t_{ff} - \frac{1}{2} g \, t_{ff}^2 \tag{2}$$

where x_o, z_o, $\dot{x}_o$, and $\dot{z}_o$ are BO positions and velocities, t_{ff} is the time of free flight, and g is gravitational acceleration equal to 32.2 fps^2. The problem can be further simplified if it is assumed that $z = z_o$ and that x_o is small compared to x. Hence

$$x \cong \dot{x}_o t_{ff} \tag{3}$$

$$0 \cong \dot{z}_o t_{ff} - \frac{1}{2} g \, t_{ff}^2 \tag{4}$$

From (3) and (4) it follows that

$$t_{ff} \cong + \frac{2 \dot{z}_o}{g} \tag{5}$$

and

$$x \cong + \frac{2 \dot{x}_o \dot{z}_o}{g} \tag{6}$$

or in polar form

$$x \cong + \frac{2 v_o^2 \sin 2\gamma}{g} \tag{7}$$

From (6) the velocity miss coefficients in Table 2 are easily derived. These coefficients relate burnout velocity errors to impact position errors.

Table 2 - MISS COEFFICIENTS FOR V-2 TRAJECTORY

Coefficient	Expression	Numerical Value for $V_o = 6000$, $\gamma_o = 54°$
$\frac{\partial x}{\partial \dot{x}}$	$+ \frac{2\dot{z}_o}{g}$	$0.060 \frac{mi}{fps}$
$\frac{\partial x}{\partial \dot{z}}$	$+ \frac{2\dot{x}_o}{g}$	$0.044 \frac{mi}{fps}$

The effect of a typical perturbation will now be examined. Consider the case where the thrust is 5% high during the entire 70 seconds of powered flight as shown by the upper path in Figure 3. The velocity at burnout is actually the
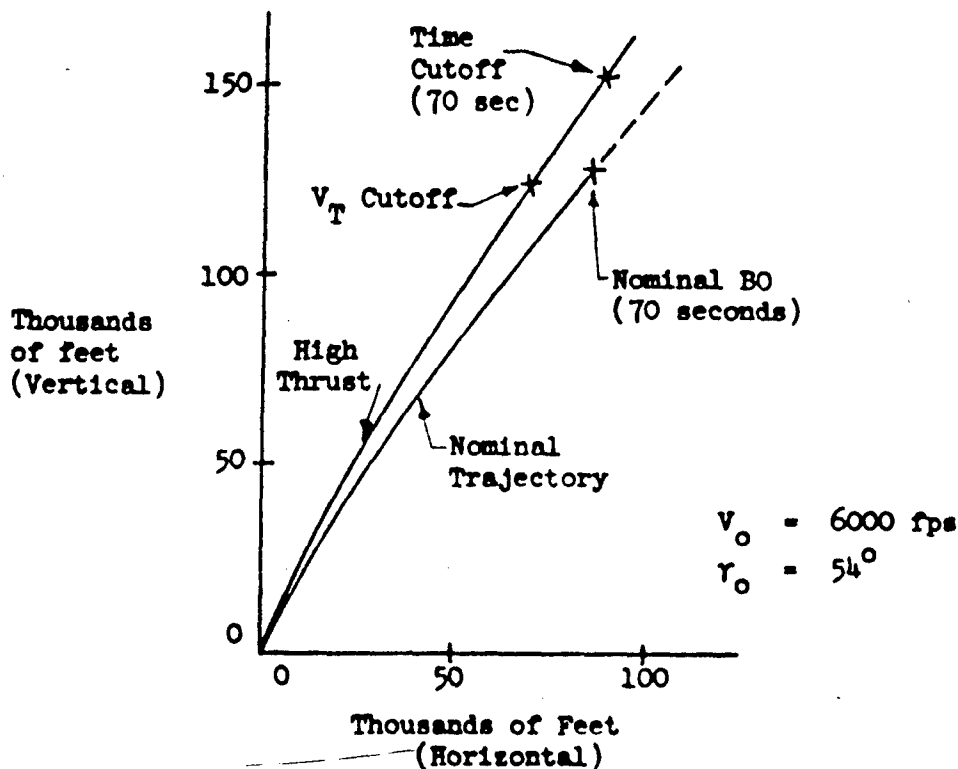

Figure 3 - V-2 POWERED FLIGHT FOR NOMINAL AND HIGH THRUST

integrated effects of both thrust and gravity so that

$$\dot{x}_o = \dot{x}_{TO} = 3500 \qquad (8)$$

$$\dot{z}_o = \dot{z}_{TO} + \dot{z}_{go} = 4800 \qquad (9)$$

where the "T" subscript indicates thrust and the "g" subscript gravity.

For a 70 second powered flight, $\dot{z}_{go}$ is equal to 2254 fps, assuming g to be a constant. Hence

$$\Delta\dot{x}_0 = \Delta\dot{x}_{TO} = .05(3500) = 175 \text{ fps} \tag{10}$$

$$\Delta\dot{z}_0 = \Delta\dot{z}_{TO} = .05(4800 + 2254) = 353 \text{ fps} \tag{11}$$

Using the miss coefficients from Table 2, Δx can be computed to be

$$\Delta x = .06(175) + .044(353) = 26.0 \text{ miles} \tag{12}$$

Clearly a downrange miss of this size would make the system unacceptable for most applications.

A More Sophisticated System - If the system is to perform adequately in the face of expected perturbations, then it appears that some closed loop scheme must be employed. The Germans took the first step in this direction by mounting an integrating accelerometer along the V-2's roll axis. Thrust was then terminated when a pre-set value of thrust velocity was reached. Again consider the case of a missile with thrust 5% high. Referring to (8) and (9) it is seen that the only BO velocity perturbation will be due to $\dot{z}_{go}$ which will be 5% low in magnitude since the time to BO is reduced by this amount and g has been assumed constant. The reduction in time, of course, is due to the fact that the pre-set value of V_{TO} will be reached earlier due to the high thrust. The downrange miss is now computed to be

$$\begin{aligned}\Delta x &= .044 \; \Delta\dot{z}_0 \\ &= .044 \; (.05)(2254) = 5.0 \text{ miles}\end{aligned} \tag{13}$$

Comparing (13) and (12) shows that the downrange miss has been reduced by a factor of five. A further reduction could probably be achieved by cutting off as a function of both time and thrust velocity. Beyond this, path variations in gravity and non-standard burnout position would have to be considered. Also

attention would have to be paid to the lateral direction where winds and mis-
alignments can cause substantial errors. In Section 2, a more basic approach
to the guidance problem is taken, and the basic elements of a system are dis-
cussed.

2.0 ELEMENTS OF GUIDANCE

In the previous section, it was shown that a considerable improvement in system accuracy can be obtained with a relatively simple guidance system employing a single body mounted accelerometer. It is easily shown, however, that the future trajectory of a vehicle does not depend on any single variable, but on all six components of its present position and velocity (and time also in the case of a moving target). It is therefore obvious that in general any really precise guidance system must measure and utilize (at least implicitly) all seven variables. For purposes of analysis, the system can be divided into three functions

  (1) Navigation

  (2) Computation

  (3) Control

In the paragraphs that follow, each of these functions, as well as their combined capability, is examined in detail. Inertial components are assumed to be perfect, since component error effects can be analyzed separately.

2.1 The Navigation Function*

Navigation, as defined here, consists of determining the vehicles position and velocity in some known frame of reference. Related to this is the computational coordinate system, which can be either inertial or earth fixed. It can be of the local vertical type, but usually is not. Two of the more common coordinate systems are

  (1) Launch-Centered Inertial - The system is inertial and is centered at the launch site at the instant of launch. It typically has x horizontal and in the launch direction, z vertical, and y completing the right handed set. It may be desirable from a computational standpoint to rotate x and z somewhat in the x-z plane.

  (2) Launch Centered Earth Fixed - This is an earth fixed coordinate system, having the same original orientation as (1). It is used primarily when it is desirable, for hardware reasons, not to remove earth rate torquing from the gyros at launch. Computationally it has both advantages and disadvantages to be discussed later.

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*It is possible to in effect combine the navigation and computation functions into a single operation. Such schemes are not covered here.

Any actual inertial guidance system has up to three other coordinate systems of interest. All four have the same center, but may have different orientations. First there is the platform coordinate system determined by the leveling and alignment references. Secondly there is the gyro coordinate system. Finally there is the accelerometer coordinate system. Originally there was only a single system, but schemes presently being used to reduce component errors and ~~simplify~~ computations frequently require all four. From a computational standpoint, the only ones of interest are the accelerometer and computational coordinate systems, since it is necessary that the airborne computer mechanize the appropriate transformation matrix.

The Navigation Loop - Since accelerometers measure only non-gravitational forces, the total acceleration is given by

$$\ddot{\bar{R}}(t) = \bar{g}(\bar{R}) + \bar{a}_T(t) \tag{14}^*$$

where $\bar{R}$ is inertial position, $\bar{a}_T$ is thrust or measured acceleration, and $\bar{g}$ is gravitational acceleration. The block diagram for a navigational or kinematic loop is shown in Figure 4. As can be seen, sensed acceleration is rotated into the computational coordinate system and is then added to the
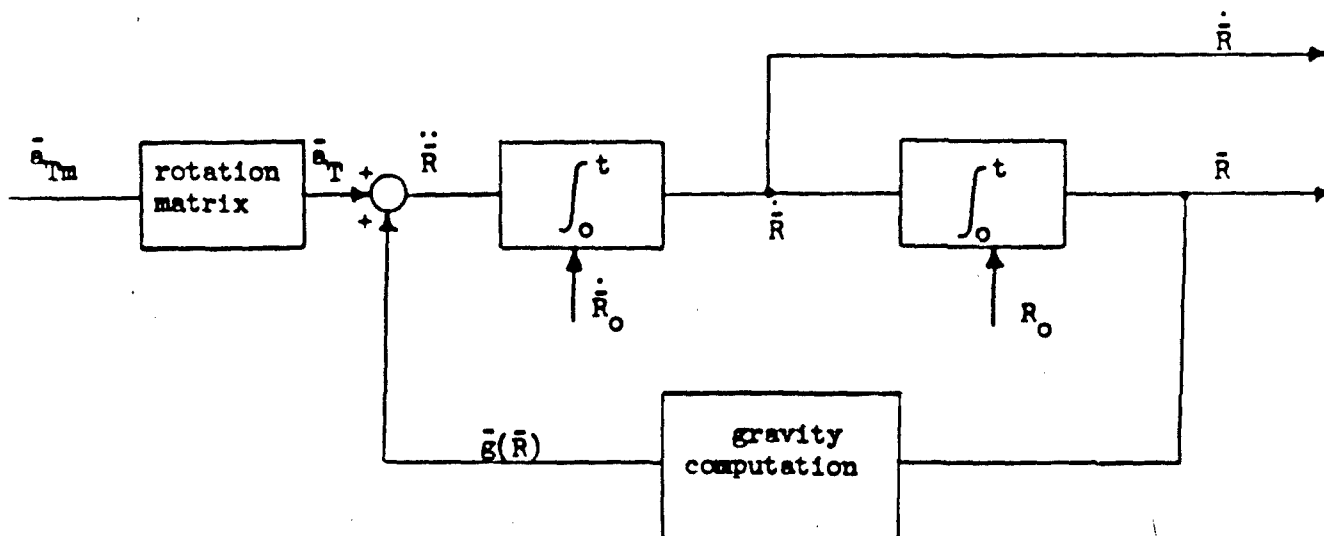

Figure 4 - THE NAVIGATION LOOP

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*A bar over a variable, as in $\bar{g}$, indicates a vector quantity.

gravitational acceleration which has been computed based on the computed position.
Position, of course, is obtained by doubly integrating the total acceleration.
All but the actual sensing of acceleration is usually accomplished in the computer,
in most cases a digital computer. If, as is frequently the case, the accelerometer
actually generates pulses representing velocity increments, the computer accumu-
lates the increments continuously and periodically adds the total to the integrated
gravitational acceleration. When accuracy permits an analog computer can be used.
Here the separation between sensors and computer may be rather obscure.

The presence of the gravity computation is one of the basic features of a precise
inertial guidance system which distinguishes it from the more rudimentary variety
like the one described in Section 1. If all trajectories were close to nominal,
there would actually be no need for a gravity computation, since the effect of
gravity could be pre-calculated. Non-standard missiles, launch delays with a
non-earth-fixed target, winds, etc., will, however, cause non-standard gravitational
accelerations. This is particularly true of long range missiles where both the
magnitude and direction of $\bar{g}$ can change appreciably during the long powered
flight period.

The Gravity Computation - The basic expression for gravity assuming a
round earth is

$$\bar{g} = - \frac{GM}{|\bar{R}|^3} \bar{R} \tag{15}$$

where $\bar{R}$ equals the position vector measured from the center of the earth, G
is the universal gravitational constant, and M is the mass of the earth. If
oblateness is considered, as it must be in most cases[*], the formula contains addi-
tional terms. From a mechanization standpoint, there are two objections to equa-
tion (15). First the coordinate system is unnatural for an inertial system;
second, and most important, the expression contains a division and a square root,
both of which are relatively slow processes in airborne digital computers.

The above problem can often be circumvented by observing that in many cases the
gravity expression need be valid over only a few miles in y and a few
hundred miles in x and z . If such is the case, (15) can be expanded in

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[*] If not considered, oblateness can cause a miss in the order of 10 n.mi. for
a 5500 n.mi. ICBM.

a Taylor series about a point midway along the powered flight trajectory,[*] an example being

$$\bar{g} = (\bar{R} + \bar{R}_o)\, GM\, [C_o + C_1 x + C_2 z + C_3 x^2 + C_4 z^2 + C_5 xz] \qquad (16)$$

Like any other power series, the number of terms required depends on the variations in $\bar{R}$ and the accuracy desired. It can be shown that all of the coefficients are functions of the launch latitude, launch azimuth, and expansion point. It is possible also to include the effect of centripetal acceleration in the expression when an earth fixed coordinate system is used.

Dynamic Behavior - Some insight into the dynamic behavior of the navigation loop can be obtained by writing the components of (15) in terms of a rectangular coordinate system with its origin in the vicinity of the trajectory and its z axis vertical. Assuming motion in the trajectory plane, only g_x and g_z will be considered. These can be written

$$g_x = \frac{-\, GM\, x}{[x^2 + (R_o + z)^2]^{3/2}} \qquad (17)$$

$$g_z = \frac{-\, GM(z + R_o)}{[x^2 + (R_o + z)^2]^{3/2}} \qquad (18)$$

where it is easily shown that GM is equal to $g_o R_o^2$. Expanding equation (14) into component form gives

$$\ddot{x} = a_{Tx} + g_x \qquad (19)$$

$$\ddot{z} = a_{Tz} + g_z \qquad (20)$$

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[*] See for example, "A Polynomial Gravity Expansion for the Arma Inertial Guidance System", (Confidential), by D. W. Whitcombe, Space Technology Laboratories Report GM 41.2-37, 17 December 1956.

Writing the perturbation equations for (19) and (20) and assuming that gravity perturbations are due to position perturbations gives

$$\Delta\ddot{x} = \Delta a_{Tx} + \frac{\partial g_x}{\partial x} \Delta x + \frac{\partial g_x}{\partial z} \Delta z \tag{21}$$

$$\Delta\ddot{z} = \Delta a_{Tz} + \frac{\partial g_z}{\partial x} \Delta x + \frac{\partial g_z}{\partial z} \Delta z \tag{22}$$

Taking the partial derivatives of (17) and (18) in the vicinity of the origin and substituting them into (21) and (22) gives the perturbation equations for the navigation loop

$$\Delta\ddot{x} + \frac{g_0}{R_0} \Delta x = \Delta a_{Tx} \tag{23}$$

$$\Delta\ddot{z} - \frac{2g_0}{R_0} \Delta z = \Delta a_{Tz} \tag{24}$$

The solutions to (23) and (24) for constant values of Δa_T perturbations are

$$\Delta x = \frac{\Delta a_{Tx}}{g_0/R_0} [1 - \cos\sqrt{\frac{g_0}{R_0}}\, t] \tag{25}$$

$$\Delta z = \frac{\Delta a_{Tz}}{2g_0/R_0} [\cosh\sqrt{\frac{2g_0}{R_0}}\, t - 1] \tag{26}$$

Near the surface of the earth, the sinusoidal oscillation in x has the familiar Schuler frequency where $2\pi\sqrt{R_0/g_0} \approx 84$ minutes. The z channel is, of course, divergent with time. For the three dimensional case, the behavior of the y channel is identical to that of the x channel. The point to be brought out is that the computational loop is "Schuler tuned" even though the platform is not

locally leveled. Since, however, rocket vehicles invariably have powered flight times much shorter than 84 minutes, acceleration errors propagate into velocity and position errors proportional to time and time squared; from a perturbation standpoint, the gravity loop appears to be open for time intervals small compared to the Schuler period.

Multiple Guidance Periods - When inertial guidance is to be used for the launching of high altitude earth satellites or for space travel, additional powered flight periods can occur after long periods of free flight. In order to establish the initial conditions for the navigation loop at the beginning of a subsequent powered flight period, it is of course, possible to operate the navigation loop during the free flight period. If this is done, (1) accelerometers must be disabled, since any output during the free flight period is erroneous and should be ignored, (2) the gravity formula or expansion must be valid over the entire free flight region, and (3) the digital computer may consume considerable power over a long coast.

A second method for establishing the initial conditions for the second powered flight or burn is to predict them from the observed first burnout position and velocity. Since the guidance equations can control four of the initial conditions,[*] it is necessary to predict only the remaining three. The prediction approach, which can be accomplished with either explicit formulas or polynomial expansions, has the disadvantage of requiring more computer memory. All things considered, though, it is usually the preferred approach for free flights of any appreciable duration.

2.2 The Computation Function

Given the position and velocity of the vehicle as well as the time, the computation function consists of utilizing this information to generate error or status signals which can be used to control the flight path of the rocket in a way which will result in the missions being accomplished. The functions used to generate the control error (or status) signals are commonly called "guidance equations".

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[*]This is shown in Section 2.2.

Thrust Magnitude Control - Rocket engines whose thrust can be controlled in
magnitude as well as direction are said to have thrust magnitude control. Vehicles
with this feature have more flexibility. For example, they can be made to fly the
exact nominal trajectory and hence can be made to burnout at a pre-specified posi-
tion, velocity, and time. The guidance computations in this case can be greatly
simplified, since it is only necessary to measure the three components of thrust
acceleration and to compare them or their time derivative with the nominal profiles
which have been stored as functions in the airborne computer. By controlling the
direction (steering) and the magnitude (throttling) of the thrust it is possible
to match the stored profiles to an arbitrary degree, depending only on the response
of the control system.

A typical configuration for a system with thrust magnitude control is shown in
Figure 5. Besides being simple to mechanize, such a system can be advantageous
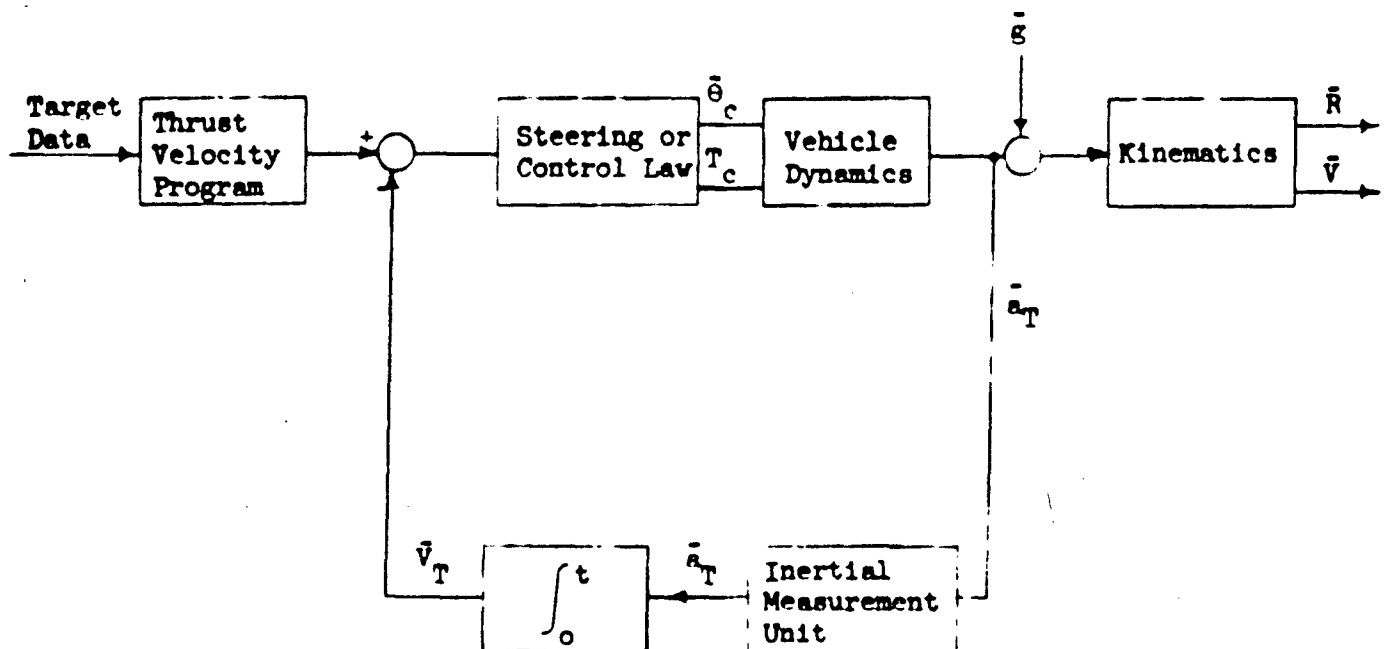
Figure 5 - A GUIDANCE SYSTEM WITH THRUST CONTROL

when it is necessary to match a desired trajectory exactly, as in the case with
the rendezvous problem. It should be pointed out, however, that the combination
of launch time variations and a non-earth-fixed target causes the nominal trajectory
concept to loose most of its meaning. Throttling in any case has not been con-
sidered desirable from a propulsion hardware standpoint, so very few rocket engines
have this capability. The remainder of this chapter is therefore concerned only
with the class of vehicles whose thrust can be controlled in direction only, and
whose thrust magnitude varies in the order of a few percent from engine to engine.

Available Techniques - Most of the available techniques for guiding rocket vehicles, were developed as a part of the various ballistic missile programs. Since these vehicles are intended to deliver their payloads to a precise point on the surface of the earth, the first approach to their guidance was to control the missiles flight p.th on the basis of predicted errors in impact. Functions of the form

$$M_d = f_d(\bar{R}, \dot{\bar{R}}, t) \tag{27}$$

$$M_c = f_c(\bar{R}, \dot{\bar{R}}, t) \tag{28}$$

can be generated, where M_d is the downrange miss and M_c is the crossrange miss, both at the target altitude. The functions will differ slightly with the miss coordinate system employed. The one most often used is the so called instantaneous impact point (IIP) coordinate system. This is an orthogonal coordinate system centered at the target, with the M_d axis defined by the downrange IIP locus or the locus of impact points which occurs when only the thrust cutoff time of a nominal vehicle is varied.

Impact error in the actual system is controlled by changing the flight path (steering) in the lateral direction (yaw) until M_c is driven to zero and then terminating thrust when M_d reaches zero. For this case, pitch steering is not required, since only two degrees of control are necessary. Here an open loop pitch attitude program can be used, e.g., constant attitude.

While the above scheme is workable, it has several disadvantages. First of all the M_d and M_c expressions are both functions of seven variables and are laborious to compute. Probably even more important is the fact that the expressions are not general and are awkward for anything but ballistic missile (ICBM, IRBM, etc.) applications. Hence, the "required velocity" concept is now more generally used[*].

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[*] Much of the material on required velocity is extracted from, "An Introduction to Inertial Guidance Concepts for Ballistic Missiles", (Tutorial Report) by David W. Whitcombe, Space Technology Laboratories, 12 April 1959.

Required Velocity - This concept is based on the fact that at each space-time point in the powered flight region, a required velocity vector

$$\bar{V}_R = \bar{V}_R(\bar{R}, t) \tag{29}$$

may be defined and computed such that the resulting free flight trajectory will satisfy four general guidance constraints.

Consider for example the ICBM trajectories shown in Figure 6. Although A1 and A2 arrive at point A at different times, all three trajectories can be made to "hit" the target. That is, after a period of time, t_{ff} (the time of free flight), the three trajectories satisfy the three guidance constraints that $\bar{R}$ be equal to $\bar{R}_T$. Consider, however, the two trajectories originating at
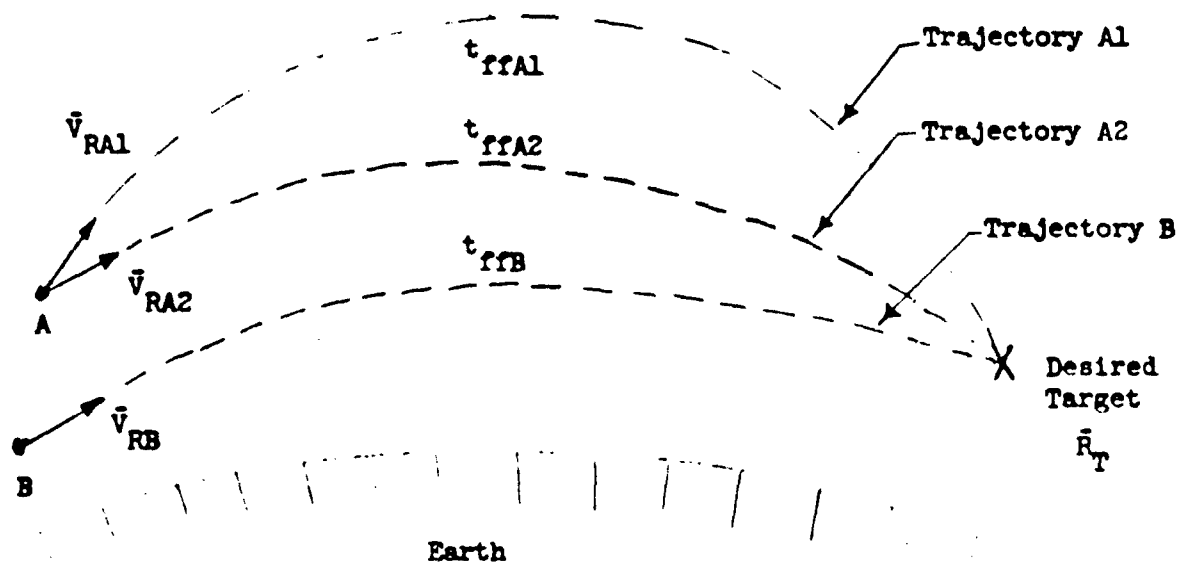

Figure 6 - THREE FREE FLIGHT TRAJECTORIES, WHICH
ALL HIT THE DESIRED TARGET

point A. In order to specify the required velocity at A , an additional guidance constraint must be imposed. This should be obvious since it pre-viously has been shown that only two degrees of freedom are required to

control impact and $\bar{V}_R$ naturally has three components. For example, it is
possible to hold the total time of flight from launch constant. This has
the effect of making an earth fixed target stand still, which sometimes has
certain advantages from a computational standpoint. For this set of con-
straints, the missile will reach the target with an arbitrary velocity which
depends on the particular burnout conditions. Some of the other possible
constraints are

1) Burnout velocity magnitude

2) Burnout velocity elevation angle

3) Any component of burnout velocity

4) Burnout energy

5) Burnout angular momentum magnitude

6) Velocity magnitude at the target

7) Velocity elevation angle at the target

8) Any component of velocity at the target

9) Time of free flight.

Some Examples - It is not necessary to include all three target position
coordinates as constraints. For one satellite launch problem it was found
desirable to use the following constraint:

1) $x = x_T$

2) $z = z_T$

3) $\dot{y} = \dot{y}_T$

4) constant total time of flight.

Here t_{ff}, $\dot{x}_T$, $\dot{z}_T$, and y_T are all allowed to vary. In the case of satellites
which have a number of thrust periods separated by coast periods, it is usually
desirable to use different sets of constraints during the various guidance periods.

The generality of the required velocity concept is illustrated by the fact that
the simple case of steering a vehicle to a particular velocity at burnout is
included. Here the constraints are

1) $\dot{x} = \dot{x}_T$

2) $\dot{y} = \dot{y}_T$

3) $\dot{z} = \dot{z}_T$

4) $t_{ff} = 0$

In this case, the target position which reduces to the burnout position will vary depending on the propulsion system, winds, etc.

The above examples have illustrated that all essential guidance information is included in the specification of the required velocity vector. If the missile is steered in both pitch and yaw, it is theoretically capable of achieving exactly the required velocity when thrust is terminated, and hence capable of satisfying the four guidance constraints.* Since satisfying four constraints, actually means causing three of the constraints to occur simultaneously with the natural occurance of the fourth, it is reasonable that this degree of per- formance should be attainable with the three degree of control available in most rocket vehicles, i.e., pitch steering, yaw steering, and thrust termination. As was pointed out in an earlier paragraph, it is possible to satisfy all seven constraints if the rocket engine is throttleable.

Another Approach - Since only three constraints are actually required to cause a ballistic missile to impact at a target, it is sometimes convenient to work in terms of a two component (x and y) required velocity vector. In this case $\dot{z}$ is allowed to be arbitrary and $\tilde{V}_R$ becomes a function of $\dot{z}$ as well as position and time. Hence

$$\tilde{V}_R = \tilde{V}_R(\bar{R}, t, \dot{z}) \tag{30}$$

Since $\dot{z}$ is arbitrary pitch steering is not required for guidance purposes, it may be used to satisfy antenna look angle constraints or may be eliminated altogether in the interest of simplicity.

Some of the presently used methods for computing required velocity are described in Section 3.

2.3 The Control Function

The basic function of the control portion of a rocket guidance system is to steer the vehicle in a way that will cause the actual velocity to become equal to the required velocity. When this occurs thrust is terminated and the

*The work of D. MacPherson at Space Technology Laboratories has shown that it is often possible to satisfy more than four constraints, by steering in a particular way. His work is not covered here.

guidance systems plays no further roll unless there are additional guided phases. Since the steering loops are feedback systems, it is convenient to define an error signal, the most commonly used one being given by the expression

$$\bar{V}_g = \bar{V}_R - \bar{V} \tag{31}$$

where $\bar{V}$ is equal to $\dot{R}$, and where $\bar{V}_g$ is called the velocity-to-be-gained or alternatively the velocity-to-go. It is the function of the steering loop to drive $\bar{V}_g$ to zero; the steering loop is actually a velocity control system[*] with $\bar{V}_R$ as the commanded input.

The Steering Loops - A block diagram of a high quality inertial guidance system for rocket vehicles is shown in Figure 7. As can be seen, both the navigation loop and the guidance equations are integral parts of the steering loop. The velocity control law (or function) must be chosen so that some time will always exist where all components of $\bar{V}_g$ are equal to zero. Since the system is non-linear and time-varying, the development of the proper function is not a simple task. While ballistic guidance systems are usually thought of as velocity control systems, as stated above, an outer loop does exist. The predicted values of the three or four target variables of interest are constantly being compared with their specified values. This is, of course, done implicitly by the guidance equations. The result of the comparison is the command signal for the velocity loop. In practice $\bar{V}_R$ changes much slower than $\bar{V}$, so for stability studies, the former can be treated as a forcing function.

As was pointed out in Section 1.2, the relatively fragile nature of long range rocket vehicles, usually requires that the steering be divided into two phases: (1) an atmospheric phase and (2) a vacuum phase. During the first phase, which corresponds to the region where aerodynamic effects are appreciable and where the nominal trajectory is a zero lift, or gravity, turn, only a rather "gentle" type of steering can be tolerated. Any violent maneuvering will result in excessive structural loads which may cause the vehicles destruction.

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[*]The term control system is frequently taken to mean attitude control system. In this chapter, "control" has a more general meaning. When attitude control is meant, it will be so specified.
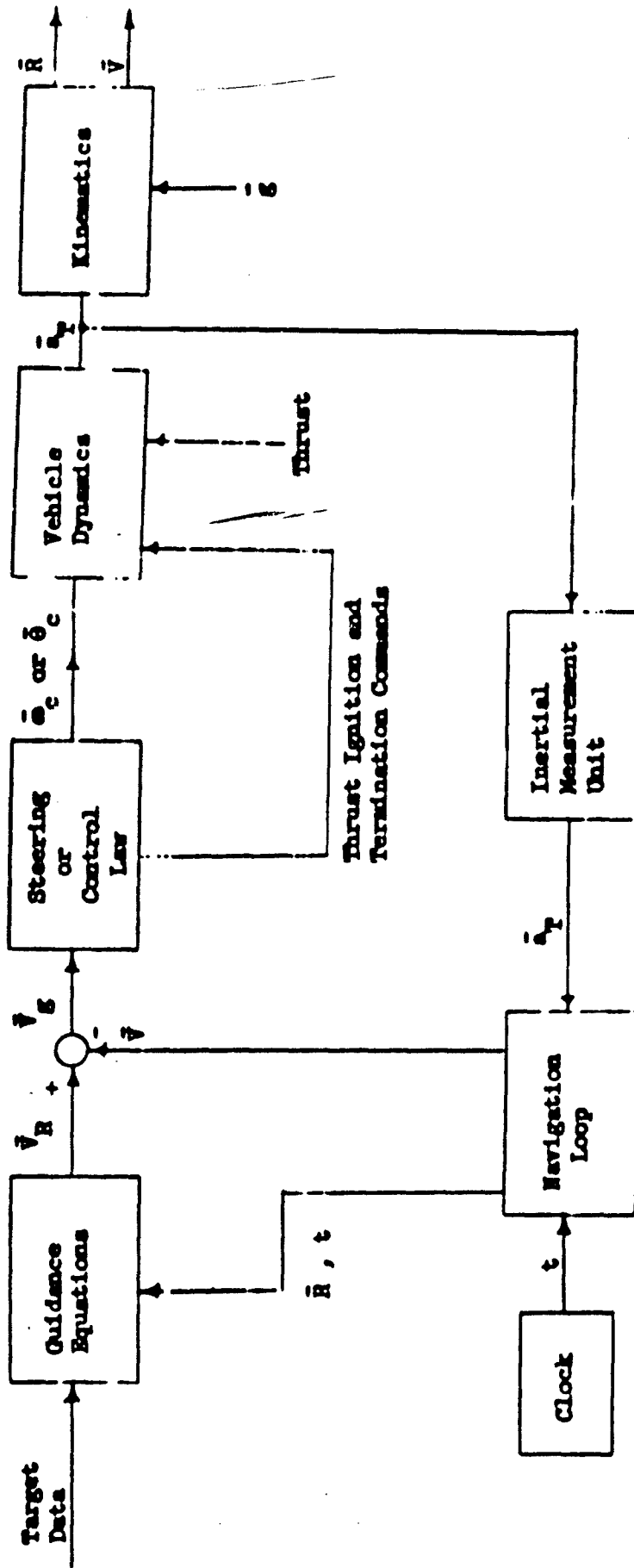
Figure 7 — BASIC BLOCK DIAGRAM FOR AN INERTIAL GUIDANCE SYSTEM FOR A ROCKET VEHICLE

Steering Signals - The net result of the "gentle" steering requirement is that $\bar{V}_g$ is usually not an acceptable steering signal during phase 1, because of the violent maneuvering which it can cause. The two approaches which have been used most successfully to date are

- (1) Programming pitch and yaw attitude as functions of time.
- (2) Steering the missile on some moderately well behaved function such as one which commands a given velocity profile, e.g. null $\dot{y}$ with yaw control and command a functional relationship between $\dot{x}$ and $\dot{z}$ with pitch control.

Any scheme used during the atmospheric phase besides being "gentle" should keep the vehicle as close as possible to the nominal trajectory, even in the phase of perturbations, so that the required region of guidance equation validity is minimized.

During the second or vacuum steering phase, yaw attitude or attitude rate is commanded with some function of $\bar{V}_g$. Pitch attitude or attitude rate may be commanded as in the atmospheric phase or by some function of $\bar{V}_g$, depending on the number of guidance constraints to be satisfied. In any case, thrust is terminated when $\bar{V}_g$ reaches zero or nearly so. The steering problem, including a number of presently used approaches, is discussed more completely in Section 4. Regardless of the system used, excessive maneuvering should be avoided, since it is inefficient from a fuel standpoint.

This section has described a general type of rocket vehicle inertial guidance system. It was shown that the scheme has three main functions: (1) Navigation, (2) Computation, and (3) Control. Such systems can theoretically achieve almost arbitrary accuracy, limited only by the capacity of the airborne computer. In practice, of course, component errors place a limit on the achievable performance.

3.0 GUIDANCE EQUATIONS

Given the time, and the position and velocity of a vehicle, a guidance system
must be capable of generating signals which are a measure of the vehicles present
ability to accomplish its intended mission. These signals are in turn used to
modify the flight path and to terminate thrust in a way which guarantees that
all guidance constraints will be satisfied. The generation of these error or
status signals, which was referred to as the computation function in Section 2,
is accomplished through the use of guidance equations. Two of the more common
types of guidance equations are discussed in this Section; both are based on the
required-velocity concept.

In general the closed-form computation of required velocity for the case of a
rotating, oblate earth with atmosphere is not possible. Approximate calculations
of more than adequate accuracy can be accomplished, however, in a number of ways,
two of the more useful ones being
    1) Explicit Guidance Equations
    2) Delta Guidance Equations
From a practical standpoint, the methods differ in two important respects:
(1) complexity of the in-flight computations and (2) amount of targeting or
pre-computation required. Basically one can be traded for the other. Explicit
equations are rather complicated from a mechanization standpoint, but require
a minimum of pre-computation. For delta, essentially the reverse is true.
The selection for any actual system will depend mainly on the relative importance
of these two points.

3.1 Explicit Guidance Equations

The most straight forward approach to the guidance of a ballistic vehicle is
to work directly with the free flight equations of motion. For purposes of
illustration consider again the case of a missile traveling in a vacuum under
the influence of a constant gravity field (usually called a "flat earth"). The

equations of motion as previously given in Section 1 are

$$x = x_o + \dot{x}_o t_{ff} \tag{32}$$

$$z = z_o + \dot{z}_o t_{ff} - 1/2 \, g \, t_{ff}^2 \tag{33}$$

where x is downrange, z is up, and g is equal to 32.2 fps^2. The geometry is shown in Figure 8. Motions out of the trajectory plane are not considered for the present.
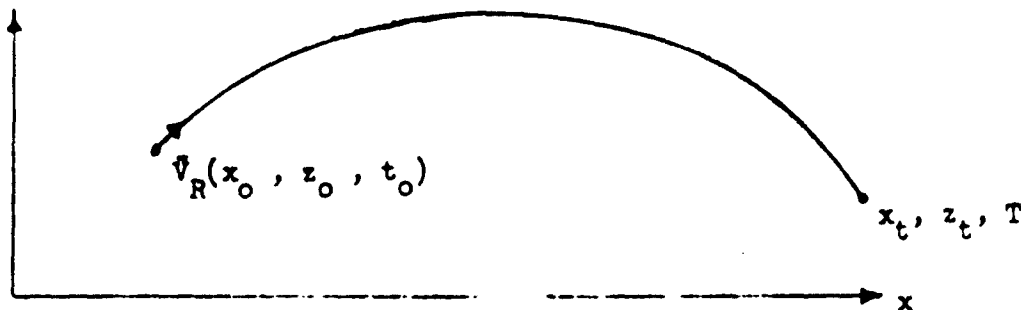
Figure 8 - FLAT EARTH EXPLICIT GUIDANCE EXAMPLE

If the total time of flight is constrained to be equal to T then the x and z components of the required velocity, $\tilde{V}_R$, are given by

$$V_{Rx} = \frac{x_t - x_o}{T - t_o} \tag{34}$$

$$V_{Rz} = \frac{z_t - z_o}{T - t_o} + 1/2 \, g(T - t_o) \tag{35}$$

where

$$t_{ff} = T - t_o \tag{36}$$

If the missile is steered in pitch so that V_z is equal to V_{Rz} and if the thrust is terminated when V_x is equal to V_{Rx}, then the missile will pass through point (x_t, z_t) at time T.

If impact time is of no interest as is sometimes the case, the time of free flight, t_{ff} , can be found from (33) as a function of z and $\dot{z}$ as given by

$$t_{ff} = \frac{\dot{z}_o + \sqrt{\dot{z}_o^2 - 2g(z_t - z_o)}}{g} \tag{37}$$

Substituting the expression for t_{ff} , (37), back into (32) gives the expression for V_{Rx}

$$V_{Rx} = \frac{g(x_t - x_o)}{\dot{z} + \sqrt{\dot{z}^2 - 2g(z_t - z_o)}} \tag{38}$$

The computation of V_{Rx} is more complicated than in the constant time of flight case, but the need for pitch steering has been eliminated, and it is now only necessary to cut-off the engine when V_x is equal to V_{Rx} . For either of these simple systems, which actually approximate the case of a short range missile, lateral motions can be handled by constraining the missile to fly in the launch-target plane, i.e., by nulling y and $\dot{y}$ by means of yaw steering. The case of a moving (but not accelerating) target is easily handled by the use of a target-fixed coordinate system. If the target is accelerated, but in a known way, the problem can still be solved, but the equations are more complex.

The Spherical Earth Case - The more important and considerably more complicated case of a missile traveling about a spherical earth will now be investigated. The effects of earth's oblateness and the re-entry atmosphere (in the case of a ballistic missile) are relatively small effects, which can be handled separately, and which will be discussed later. If the trajectory is again considered planar, the Lagrangian function for the missile in free flight is equal to

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GM\,m}{r} \tag{39}$$

where r and θ are earth centered inertial (ECI) coordinates. From (39) the equations of motion are found to be

$$\ddot{r} - r\dot{\theta}^2 + \frac{GM}{r^2} = 0 \tag{40}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{41}$$

Equation (41) may be integrated directly to give

$$\dot{\theta} = \frac{C_1}{r^2} \tag{42}$$

If the variable $p = \dot{r}$ is introduced, then

$$\ddot{r} = \frac{dp}{dt} = \frac{dr}{dt}\frac{dp}{dr} = p\frac{dp}{dr} \tag{43}$$

Substituting (42) and (43) into (40) gives

$$p\frac{dp}{dr} - \frac{C_1^2}{r^3} + \frac{GM}{r^2} = 0 \tag{44}$$

which can be integrated so that

$$\frac{p^2}{2} + \frac{C_1^2}{2r^2} - \frac{GM}{r} = C_2 \tag{45}$$

Re-arranging (45) and taking the square root of both sides leads to

$$p = \sqrt{2C_2 + \frac{2GM}{r} - \frac{C_1^2}{r^2}} \tag{46}$$

But since

$$p = \dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{C_1}{r^2} \tag{47}$$

equation (46) becomes

$$d\theta = \frac{dr}{r\sqrt{\dfrac{2C_2 r^2}{C_1^2} + \dfrac{2GM\,r}{C_1^2} - 1}} \tag{48}$$

whose integral is

$$\theta - \theta_0 = \sin^{-1}\left[\frac{1 - \dfrac{C_1^2}{GM\,r}}{\sqrt{1 + \dfrac{2C_1^2 C_2}{(GM)^2}}}\right] \tag{49}$$

Since θ_0 is arbitrary it can be set equal to zero. The equation can then be re-written so that

$$r = \frac{C_1^2}{GM\left(1 - \sqrt{1 + \dfrac{2C_1^2 C_2}{(GM)^2}}\,\sin\theta\right)} \tag{50}$$

This, however, is identical in form to the equation for an ellipse with one focus at the origin

$$r = \frac{\alpha(1 - e^2)}{1 - e\,\sin\theta} \tag{51}$$

where α is the semi-major axis and e is the eccentricity. By equating similar coefficients in (50) and (51), the constants of integration are related to the elliptical constants as follows:

$$C_1 = \sqrt{GM\,\alpha(1 - e^2)} \tag{52}$$

$$C_2 = -\frac{GM}{2\alpha} \tag{53}$$

Therefore using (42) and (53), equation (45) can be reduced to

$$\dot{r}^2 + (\dot{\theta}r)^2 = v^2 = GM\left(\frac{2}{r} - \frac{1}{\alpha}\right) \tag{54}$$

This is the so called "vis viva" integral of celestial mechanics[*] which simply shows that the speed of a body in orbit depends only on its distance from the center of force. It now becomes a problem in analytic geometry to derive an expression for the velocity required to hit a desired impact point, since (54) clearly shows it is necessary to have only the semi-major axis of the free flight ellipse joining vehicle and target.

The Required Velocity Expression - In order to derive the expression for the semi-major axis of the ellipse, it is necessary to use the following two properties of an ellipse:

(1) The sum of the distances from any point on the ellipse to the two foci is equal to the major axis of the ellipse.

(2) A line normal to an ellipse bisects the angle between the two lines to the foci.

The geometry of the situation is shown in Figure 9, where $\bar{r}$ and $\bar{r}_t$ are the missile and target vectors, ϕ is the range angle, and γ is the angle of $\bar{V}_R$ with respect to the local vertical. From the figure it can be seen that

$$x^2 + y^2 = (2\alpha - r_t)^2 \tag{55}$$

$$y = r + (2\alpha - r)\cos 2\gamma - r_t\cos\phi \tag{56}$$

$$x = (2\alpha - r)\sin 2\gamma - r_t\sin\phi \tag{57}$$

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[*]Similar derivations of the "vis viva" integral can be found in almost any book on classical mechanics, e.g., McCuskey, S. W., "An Introduction to Advanced Dynamics", Addison-Wesley, 1959.
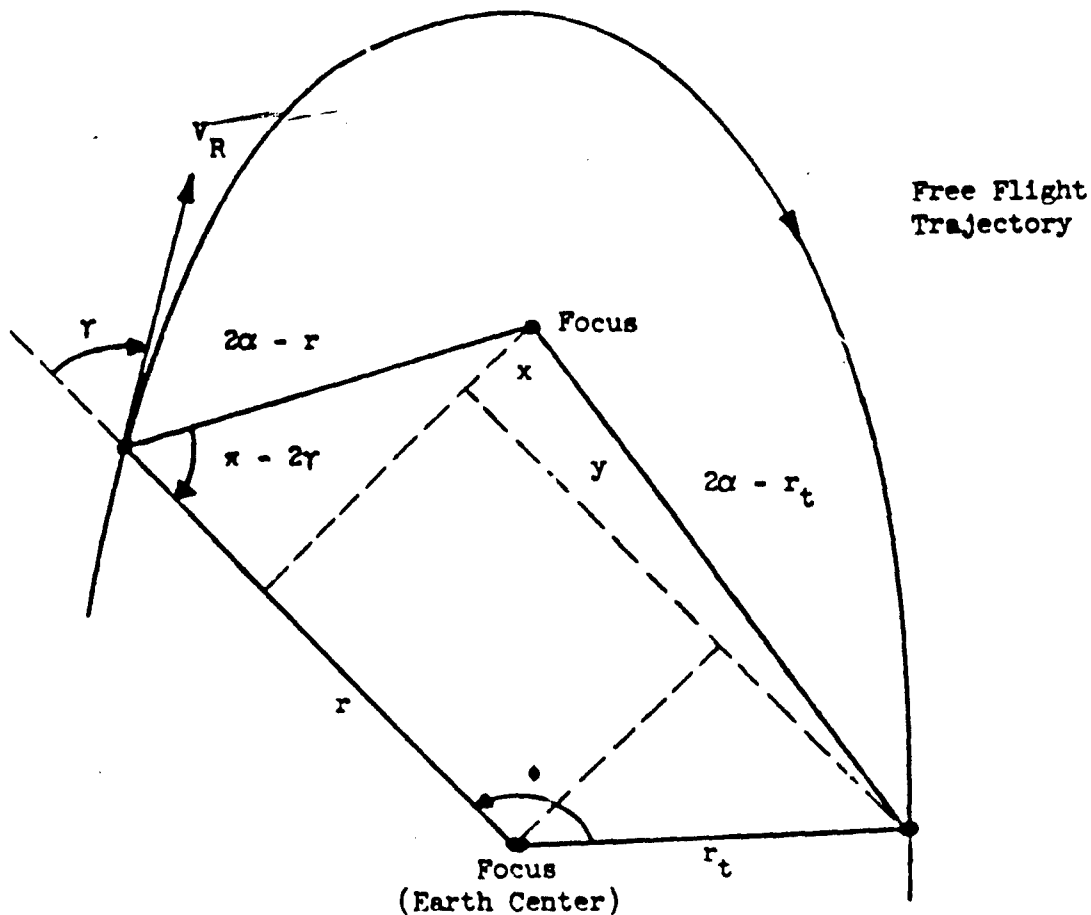
Figure 9 - GEOMETRY OF SEMI-MAJOR AXIS CALCULATION

Solving (55), (56), and (57) for α gives

$$\alpha = \frac{r}{2}\left[1 + \frac{r_t(1 - \cos\bullet)}{r - r_t - r\cos2\gamma + r_t\cos(2\gamma - \bullet)}\right] \tag{58}$$

This then is the semi-major axis of an ellipse which contains the missile and the target and is tangent to the required velocity vector.

Substitution of (58) into (54), the "vis viva" integral, gives

$$V_R^2 = \frac{2GM}{r} \frac{1 - \cos\phi}{\frac{r}{r_t}(1 - \cos 2\gamma) - \cos\phi + \cos(2\gamma - \phi)} \tag{59}$$

which is an expression for the required velocity in terms of the missile and target distances from the center of the earth, the range angle, and the velocity vector angle. It may be computationally convenient to mechanize (59) as a vector equation, replacing cosines with dot products. Thrust termination can be commanded according to either of two criteria

$$\left(\frac{V_R}{V}\right)^2 = 1 \tag{60}$$

$$V_R - V = 0 \tag{61}$$

where V_R and V are both-scalar quantities. Equation (60) has some advantages since it tends to be linear in the region of cutoff.

Effect of Earth's Rotation - Thus for an equation, (59), has been derived which gives the velocity required to hit a particular point in inertial space. Most targets, however, are not fixed in inertial space, they are usually either in orbits of their own, or they are fixed to the earth and rotate with it. In particular the ECI rectangular coordinates of an earth fixed target at some future time are given by

$$x_t = r_t \cos\theta_t \cos(\phi_t + \omega_e T_{ff}) \tag{62}$$

$$y_t = r_t \cos\theta_t \sin(\phi_t + \omega_e T_{ff}) \tag{63}$$

$$z_t = r_t \sin\theta_t \tag{64}$$

where θ_t and ϕ_t are the latitude and reference longitude of the target and where ω_e is earth's rate and T_{ff} is the time of free flight.

The above equations clearly show that in order to predict the position of the
target at impact, the time of free flight must be known. The time of free
flight, however, depends on V_R which in turn is a function of the impact
position. Hence, an iterative procedure results as follows:

1) A future target position is assumed.

2) The corresponding required velocity is computed.

3) The elements of the resulting ellipse are computed.

4) The time of flight is computed.

5) A new future target position is computed.

6) The procedure is repeated.

Because the earth moves rather slowly (1000 fps at 45° latitude), the time of
flight calculation is not very critical and the iteration can be carried out on
a cycle by cycle basis. This is not necessarily the case for targets in general.

The derivation of the expression for time of flight begins with the "vis viva"
integral, equation (54), which is equal to

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 = GM\left(\frac{2}{r} - \frac{1}{\alpha}\right) \tag{65}$$

Previously it was shown, (42) and (52), that the angular momentum of an orbit
is constant and is given by

$$r^2\dot{\theta} = C_1 = \sqrt{GM\,\alpha\,(1 - e^2)} \tag{66}$$

Squaring (66) and combining the result with (65) gives

$$\dot{r}^2 + \frac{GM\,\alpha\,(1 - e^2)}{r^2} = GM\left(\frac{2}{r} - \frac{1}{\alpha}\right) \tag{67}$$

Multiplying through by r^2/α and rearranging leads to

$$\frac{(r\dot{r})^2}{\alpha} = GM\left[e^2 - \left(1 - \frac{r}{a}\right)^2\right] \tag{68}$$

It is now convenient to introduce another frequently used orbital parameter - E , the eccentric anomaly. The geometry which defines the angle is shown in Figure 10.
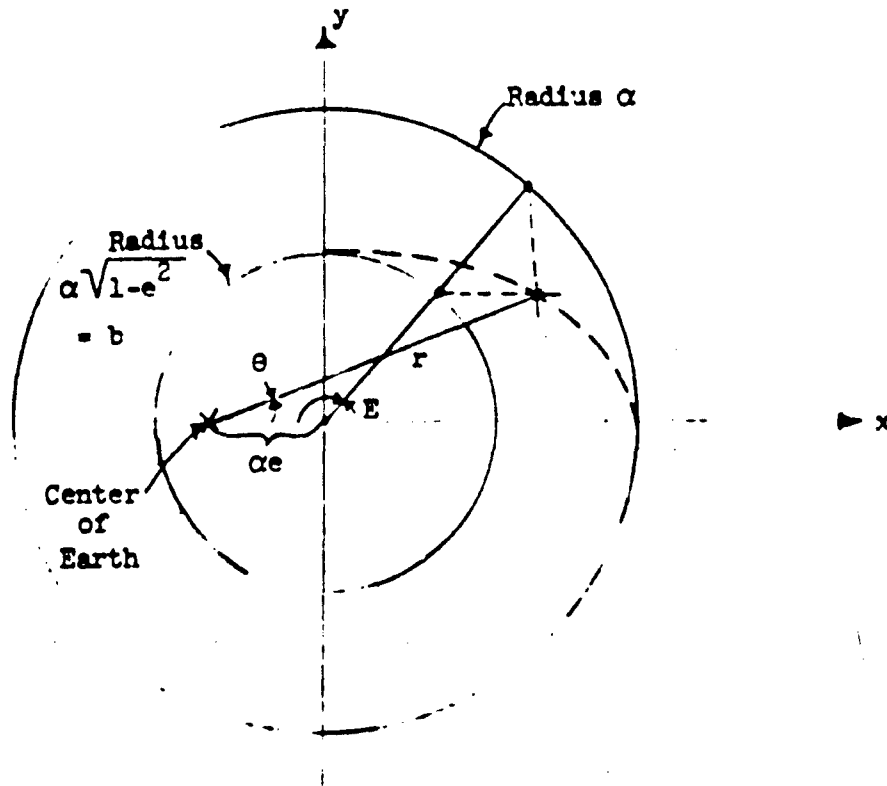
Figure 10 - ORBITAL GEOMETRY SHOWING MEAN ANOMALY ANGLE

Concentric circles of radius α and b are drawn. If a radius vector is drawn at angle E with respect to the x axis, and if parallels to the x and y axis are drawn through the points where the radius vector intersects the small and large circles, then the inter-section of the two parallels will be a point on an ellipse. The eccentric anomaly is a way of designating a particular point on the ellipse just as is θ .

From the geometry of Figure 10, it is easily shown that

$$r = \alpha (1 - e \cos E) \tag{69}$$

or that

$$e \cos E = 1 - \frac{r}{\alpha} \tag{70}$$

Therefore (68) becomes, after taking a square root,

$$\frac{r\dot{r}}{\sqrt{\alpha}} = \sqrt{GM} \; e \sin E \tag{71}$$

Multiplying (69) by its time derivative gives

$$r\dot{r} = \alpha^2 e \sin E (1 - e \cos E) \dot{E} \tag{72}$$

which can be combined with (71) to give

$$\sqrt{GM} \; \alpha^{-3/2} = \dot{E} (1 - e \cos E) \tag{73}$$

which in turn can be integrated directly to give Keplers equation

$$\alpha^{3/2}(E - e \sin E) = \sqrt{GM} \; t + C \tag{74}$$

where C is the constant of integration. The time of flight to the target follows directly from (74) and is given by

$$t_{ff} = (GM)^{-1/2} a^{3/2}[(E_t - E) - e(\sin E_t - \sin E)] \tag{75}$$

where E and E_t are the eccentric anomolies of the missile and target respectively. Since all terms of (75) are easily computed from orbital relationships, the inertial position of the target can be obtained.

Other Effects - The scheme described here is designed to hit a target moving in a known manner, such as a point on the earth or a satellite. It does not require control of radial velocity, i.e., the pitch attitude program can be arbitrary. If a more general guidance problem such as the placing of a satellite in orbit is to be solved, then radial velocity must be controlled and the equations become more complicated.

Thus far no mention has been made of ways to compensate for re-entry and oblate-
ness effects, nor have ways to handle motions out of the orbital plane been con-
sidered. The first two effects, it turns out, are relatively small and can
usually be handled by tabulated target offsets which are functions of range,
azimuth, latitude, etc. The last effect is more complicated in that some steering
is necessary. Basically all that is required is that the vehicle have no velocity
normal to the plane containing the vehicle, the target at the time of impact, and
the center of the earth. The normal velocity is given by

$$V_N = (\bar{r} \times \bar{r}_t) \cdot \bar{v} \tag{76}$$

which can be used as the error signal for yaw steering and is analogous to V_{gy} .

3.2 Delta Guidance Equations

Until very recently, the complexity of explicit equations made them very unattrac-
tive for use in inertial guidance systems. Airborne computers of reasonable size
simply were not fast enough to solve the equations in real time, e.g., once or
twice per second. It was therefore necessary to find other ways to handle the
problem.

Since, as has already been stated, vehicle perturbations are relatively small
(in the neighborhood of a few percent) it was only reasonable to think in terms
of guidance equations which are power series expansions about a nominal trajectory.

General Development - A power or Taylor series in one variable is of the form

$$f(x) = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots \tag{77}$$

where x is the variable and a is the point about which the expansion occurs.
For a function of two variables, $f(x, y)$, the series is of the form[*]

$$f(x,y) = f(a,b) + \left[(x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right] f(x,y) \Big|_{a,b}$$

$$+ \frac{1}{2!} \left[(x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^2 f(x,y) \Big|_{a,b} + \dots \tag{78}$$

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[*]See any advanced calculus book, e.g., C. R. Wylie, "Advanced Engineering Mathematics,"
pp 598-604, McGraw-Hill, 1951.

where x and y are the variables and a,b is the point about which the series is expanded. The similarity to the one variable case is unmistakable. The extension to three or more variables is equally clear.

As has been shown previous, $\bar{V}_R$ has either two or three components depending on the number of guidance constraints to be satisfied. In the more general three component case, each component is a function of four variables - x, y, z, and t. They are, of course, implicit functions of the guidance constraints themselves also.

If expansions are to be found, then an expansion point must be selected. The first approach might be to select points all along the nominal trajectory and to program them as a function of time. Coefficients would have to be determined corresponding to each expansion point and the three expansions (V_{Rx}, V_{Ry}, and V_{Rz}) would be time varying. If the number of expansion points used was very large, the airborne computer storage requirements could easily become excessive.

A little reflection on the problem, however, soon leads one to the conclusion that the expansions need to be highly accurate only in the immediate vicinity of burnout. Hence, only a single point and a single set of expansions need be used, at least for any one guidance phase. Since the nominal burnout point (x_0, y_0, z_0, t_0) is usually considered to be the most likely burnout point, (or vector) it is usually selected as the expansion point. Three expansions similar to the following V_{Rx} expression therefore result:

$$V_{Rx} = \dot{x}_0 + k_{xx}\Delta x + k_{xy}\Delta y + k_{xz}\Delta z + k_{xt}\Delta t + k_{xxx}\Delta x^2$$

$$+ k_{xxy}\Delta x \Delta y + k_{xyy}\Delta y^2 + \ldots \ldots \tag{79}$$

where $\Delta x = (x-x_0)$, $\Delta y = (y-y_0)$, etc., the delta quantities, of course, giving the equations their name. The coefficients, k_{xx}, k_{xy}, etc., are basically partial derivatives and are defined by the corresponding terms in (78). Usually linear and some second order terms are required depending on the probable size

of the "burnout box" and the accuracy required. The number of terms required
will depend also on the miss coefficients in the direction of the expansions
validity.

A Flat Earth Example - In order to illustrate the working of delta equa-
tions, a flat-earth, two-dimensional case will again be considered. The nominal
trajectory parameters are those of the V-2 missile from Section 1 and are as
listed in Table 3.

Table 3 - V-2 TRAJECTORY PARAMETERS

Parameter	Symbol	Value
Horizontal BO Position	x_o	85,000 feet
Vertical BO Position	z_o	123,700 feet
Horizontal BO Velocity	$\dot{x}_o$	3,500 fps
Vertical BO Velocity	$\dot{z}_o$	4,800 fps
Time at BO	t_o	70 seconds
Horizontal Target Position	x_t	1,212,000 feet
Vertical Target Position	z_t	0 feet
Time at Impact	T	392 seconds

The constant time of flight constraint will be employed $(T = 392)$ and both
linear and quadratic terms will be used. The coefficients can be obtained by
performing the appropriate partial differentiations on the explicit expressions
for V_{Rx} and V_{Rz} , (34) and (35). The coefficients are summarized in
Table 4. It can be seen that while there are 18 possible linear and quadratic
terms in the two expansions, only 8 are non-zero for the flat-earth, constant-
time-of-flight case. The biggest reduction is due to the absence of cross
coupling between the x and z channels.

Table 4 - LINEAR AND QUADRATIC DELTA COEFFICIENTS FOR V-2 TRAJECTORY
(Constant Time of Flight)

Coefficient	Definition	Expression	V-2 Value
k_{xx}	$\dfrac{\partial V_{Rx}}{\partial x}$	$-\dfrac{1}{T-t_o}$	-3.1×10^{-3}
k_{xz}	$\dfrac{\partial V_{Rx}}{\partial z}$	0	0
k_{xt}	$\dfrac{\partial V_{Rx}}{\partial t}$	$\dfrac{(x_t - x_o)}{(T-t_o)^2}$	10.9
k_{xxx}, k_{xxz}	$\dfrac{1}{2}\dfrac{\partial^2 V_{Rx}}{\partial x^2}, \dfrac{\partial^2 V_{Rx}}{\partial xz}$	0	0
k_{xzz}, k_{xzt}	$\dfrac{1}{2}\dfrac{\partial^2 V_{Rx}}{\partial z^2}, \dfrac{\partial^2 V_{Rx}}{\partial zt}$	0	0
k_{xxt}	$\dfrac{\partial^2 V_{Rx}}{\partial xt}$	$-\dfrac{1}{(T-t_o)^2}$	-9.6×10^{-6}
k_{xtt}	$\dfrac{1}{2}\dfrac{\partial^2 V_{Rx}}{\partial t^2}$	$\dfrac{(x_t - x_o)}{(T-t_o)^3}$	3.4×10^{-2}
k_{zx}	$\dfrac{\partial V_{Rz}}{\partial x}$	0	0
k_{zz}	$\dfrac{\partial V_{Rz}}{\partial z}$	$-\dfrac{1}{(T-t_o)}$	-3.1×10^{-3}
k_{zt}	$\dfrac{\partial V_{Rz}}{\partial t}$	$\dfrac{(z_t - z_o)}{(T-t_o)^2} - \dfrac{1}{2}g$	-17.3
k_{zxx}, k_{zxt}	$\dfrac{1}{2}\dfrac{\partial^2 V_{Rz}}{\partial x^2}, \dfrac{\partial^2 V_{Rz}}{\partial xt}$	0	0
k_{zzz}, k_{zzx}	$\dfrac{1}{2}\dfrac{\partial^2 V_{Rz}}{\partial z^2}, \dfrac{\partial^2 V_{Rz}}{\partial zx}$	0	0
k_{zzt}	$\dfrac{\partial^2 V_{Rz}}{\partial zt}$	$-\dfrac{1}{(T-t_o)^2}$	-9.6×10^{-6}
k_{ztt}	$\dfrac{1}{2}\dfrac{\partial^2 V_{Rz}}{\partial t^2}$	$\dfrac{(z_t - z_o)}{(T-t_o)^3}$	-3.7×10^{-3}

Using the V-2 parameters, the V_{Rx} and V_{Rz} expansions become

$$V_{Rx} = 3500 - 3.1 \times 10^{-3}(x - 85,000) + 10.9 (t - 70) \qquad (80)$$

$$- 9.6 \times 10^{-6}(x - 85,000)(t - 70) + 3.4 \times 10^{-2} (t - 70)^2$$

$$V_{Rz} = 4800 - 3.1 \times 10^{-3} (z - 123,700) - 17.3 (t - 70) \qquad (81)$$

$$- 9.6 \times 10^{-6} (z - 123,700)(t - 70) - 3.7 \times 10^{-2} (t - 70)^2$$

Within the airborne computer, it is usually convenient to group the terms some-what differently, lumping the constants all into a single term.

In order to give some idea of the accuracy of the equations as well as the relative importance of the various terms, the equations have been evaluated for three sets of perturbations, one of which corresponds to the launch point. The results are summarized in Table 5. The launch point, of course, is not an expected burnout point and is only included for the purpose of placing some sort of an upper bound on equation error. For the other randomly selected perturbations (which are abnormally large by any standards), the equations per-form quite well. Even with linear terms alone the V-2 impact error would be well under a half mile if the missile were to burn out at these points. The example also graphically illustrates the relative importance of linear and higher order terms.

The More General Case - While the flat-earth example illustrates very well the functioning of delta equations, it gives a somewhat over-simplified picture of the effort that must go into obtaining the coefficients - a job usually known as "targeting". For the flat-earth case, simple formulas are available for the required velocity and the partial derivatives are rather quickly obtained. The $\bar{V}_R$ formulas themselves are simple enough to pro-bably obviate delta equations.

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*To the writer's knowledge, the first successful application of delta guidance equations to long range rockets was a result of the joint efforts of J. Caroll of American Bosch Arma Corporation and F. Baskin and T. W. Layton of Space Technology Laboratories.

Table 5

BEHAVIOR OF DELTA GUIDANCE EQUATIONS FOR V-2 TRAJECTORY

(ALL VELOCITIES IN FPS; CALCULATIONS TO NEAREST 1 FPS)

		$\Delta x = 20,000$ ft $\Delta z = 30,000$ ft $\Delta t = -10$ sec		$\Delta x = -10,000$ ft $\Delta z = -15,000$ ft $\Delta t = +10$ sec		Launch $\Delta x = -85,000$ ft $\Delta z = -123,700$ ft $\Delta t = -70$ sec	
		V_{Rx}	V_{Rz}	V_{Rx}	V_{Rz}	V_{Rx}	V_{Rz}
Constant Term		3500	4800	3500	4800	3500	4800
Δx "		-62	0	31	0	263	0
Δz "		0	-93	0	46	0	383
Δt "		-109	173	109	-173	-763	1211
$\Delta x \Delta t$ "		2	0	1	0	-57	0
$\Delta z \Delta t$ "		0	3	0	1	0	-83
$(\Delta t)^2$ "		3	0	3	0	166	-18
$\bar{V}_R$ (Linear Terms)		3329	4880	3640	4674	3000	6394
$\bar{V}_R$ (L and Q Terms)		3334	4883	3644	4674	3109	6293
$\bar{V}_R$ (From Formula)		3333	4882	3644	4675	3092	6311
Error (Linear)		-4	-2	-4	-1	-92	83
Error (L and Q)		+1	+1	0	0	+17	-18
Impact Error for L and Q Delta	Miles	0.10		0		0.62	
	Seconds	0.06		0		-1.1	

When a rotating, oblate earth with re-entry atmosphere is considered, the picture changes. While explicit equations for a spherical earth without atmosphere are presented in Section 3.1, it should be pointed out that even these approximate equations are iterative in nature and can not be readily differentiated in order to obtain delta coefficients. The procedure which has evolved to generate delta constants basically consists of the following two steps:

(1) The required velocity (V_{Rx}, V_{Ry}, and V_{Rz}) corresponding to each of a large number of x, y, z, and t perturbations is found by using iterative procedures on a digital computer free flight simulation. The simulation is a very precise model of a rotating, oblate earth with atmosphere. The points are selected in a way which makes them correspond as closely as possible to actual, realizable powered flight burnout points. The $\bar{V}_R$ will, of course, be a function of the constraints used.

(2) The empirically generated $\bar{V}_R$ data and the corresponding perturbations are used to generate the delta coefficients by means of least square fitting procedures.

The procedure for generating each of the three expansions is the same as the one for V_{Rx} which is summarized by —

$$\sum_{k=1}^{N} \left\{ \sum_{j=1}^{n} k_{xj} \Delta J(p_1^{(k)}, \ldots, p_4^{(k)}) - \Delta V_{Rx}(p_1^{(k)}, \ldots, p_4^{(k)}) \right\}^2 \to \text{Min} \quad (82)$$

where

N = number of data points

$p_1^{(k)}, \ldots, p_4^{(k)}$ = Δx, Δy, Δz, and Δt from the kth data point.

ΔJ = functions of $(p_1, \ldots, p_4)$ like Δx, Δy, $\Delta x \Delta y$, etc.

k_{xj} = coefficients of ΔJ in the V_{Rx} expansion

n = number of delta terms in ΔV_R

ΔV_{Rx} = $V_{Rx} - \dot{x}_o$

The functions, ΔJ, and the number of terms, n, are chosen to obtain the required degree of accuracy. It usually turns out that n is somewhat less for V_{Ry} and V_{Rz} than it is for V_{Rx}. The actual differences depend mainly on the coordinate system and the constraints employed.

The linear $\bar{V}_R$ coefficients can also be obtained from the linear target miss expansions. Since these expansions are more easily obtained (no free flight iterations are required), the procedure is sometimes used when only the linear coefficients are required or when it is desirable to simplify the fitting process. Consider for simplicity the two dimensional case. The linear miss expansions are of the form

$$M_x = \frac{\partial M_x}{\partial x} \Delta x + \frac{\partial M_x}{\partial z} \Delta z + \frac{\partial M_x}{\partial \dot{x}} \Delta \dot{x} + \frac{\partial M_x}{\partial \dot{z}} \Delta \dot{z} + \frac{\partial M_x}{\partial t} \Delta t \qquad (83)$$

$$M_z = \frac{\partial M_z}{\partial x} \Delta x + \frac{\partial M_z}{\partial z} \Delta z + \frac{\partial M_z}{\partial \dot{x}} \Delta \dot{x} + \frac{\partial M_z}{\partial \dot{z}} \Delta \dot{z} + \frac{\partial M_z}{\partial t} \Delta t \qquad (84)$$

where M_x and M_z are the x and z deviations from x_t, z_t at time equal to T (or as any other third constraint is satisfied). If the misses, $\bar{M}$, are set to zero, then the resulting two linear equations can be solved simultaneously for $\Delta \dot{x}$ and $\Delta \dot{z}$. The resulting expansions in Δx, Δz, and Δt will be identically the same as the linear V_{Rx} and V_{Rz} expansions. As an example, the Δx coefficient in the V_{Rx} expansion is

$$\frac{\partial V_{Rx}}{\partial x} = \frac{\left(\dfrac{\partial M_z}{\partial x} \dfrac{\partial M_x}{\partial \dot{z}} - \dfrac{\partial M_x}{\partial x} \dfrac{\partial M_z}{\partial \dot{z}} \right)}{\left(\dfrac{\partial M_x}{\partial \dot{x}} \dfrac{\partial M_z}{\partial \dot{z}} - \dfrac{\partial M_z}{\partial \dot{x}} \dfrac{\partial M_x}{\partial \dot{z}} \right)} \qquad (85)$$

The results are similar in the three-dimensional case, except that numerator and denominator contain six triple products, rather than two double products. Numerically, the computations are trivial in either case.

Summary - Explicit and delta guidance equations have been described and examples of each have been given. While these are not the only guidance equations currently being used, they do typify the two larger classes into which all guidance equations might be grouped, namely total equations and perturbation equations. Total equations require less pre-computation and will work

for larger vehicle and environmental perturbations. Perturbation equations, however, are simpler to mechanize and are inherently more flexible, since all guidance constraints can be changed simply by changing the constants. As has been stated previously the choice will depend mainly on operational requirements. Frequently it reduces almost to a matter of personal choice.

4.0 VEHICLE STEERING

In the previous section, it was shown that methods exist for continuously
computing the required velocity for a rocket in flight. The third and final
function of the guidance system, as described in Section 2, is the control
function, i.e., the vehicle's flight path must be directed in a way which will
cause all three components of the velocity-to-be-gained ($\bar{V}_g = \bar{V}_R - \bar{V}$) to reach
zero simultaneously. For reasons developed in Section 2.3, the steering is
usually broken down into atmospheric and vacuum phases. In this section, the
steering problem is discussed in some detail and a few of the currently used
methods for each phase are described.

4.1 The Atmospheric Phase

During the atmospheric phase, the steering of a rocket vehicle is basically
"open loop", i.e., $\bar{V}_g$ is not used explicitly to control the flight path.
Instead some less violent variable such as attitude or velocity is used to
maneuver the vehicle in a way which will keep it as close as possible to the
nominal trajectory without causing excessive structural loading.

The starting point for most atmospheric steering schemes is the open loop or
reference trajectory, usually a "kick" trajectory. This non-physical trajectory
is generated on a digital computer by causing the missile to rise vertically for
some period of time, at the end of which the vehicle's attitude and velocity
vector are instantaneously rotated downward by an amount known as the "kick"
angle. From this point on until the missile is essentially out of the atmosphere,
the thrust vector is caused to be directed along the velocity vector and the
missile flies a gravity turn. Beyond the atmosphere some arbitrary attitude
program, usually a constant angle, is flown.

Since the velocity vector of a physical rocket cannot be instantaneously
"kicked", the transition from vertical rise to gravity turn actually takes place
over a period of time known as the transition turn, the length of which is

measured in tens of seconds. An angle of attack must of necessity exist during
this period, but fortunately this occurs before the dynamic pressure has not
had time to build up.

The simplest approach to steering in the atmosphere is to program vehicle
attitude or attitude rate as a function of time. Yaw attitude is held to zero,
and pitch attitude is commanded as it occurs on the "kick" trajectory. The
"kick" angle is actually caused to occur slowly over a period of a few seconds.
A typical pitch rate program is shown in Figure 11. The program, which in this
case changes in steps can be generated mechanically or electronically. The
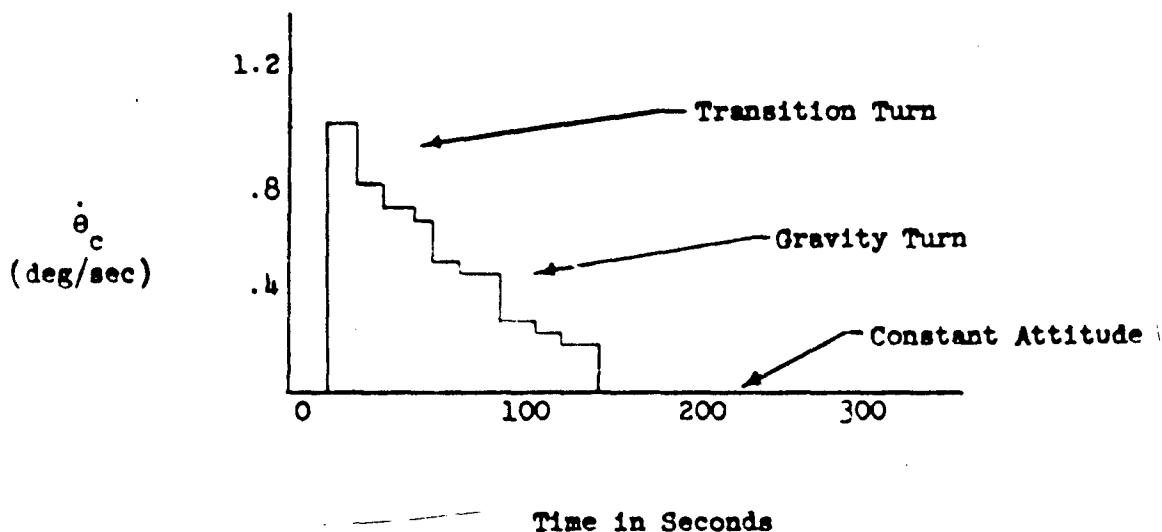

Figure 11 - TYPICAL PITCH RATE PROGRAM

corresponding pitch attitude program could also be used. In that case a more
continuous program is required, since attitude is commanded directly, without
the smoothing effect of an integration.

Velocity Steering - Somewhat better performance is obtained by steering
the missile on velocity. The velocity may or may not include integrated gravita-
tional acceleration. Consider first the case of thrust velocity alone, i.e.,
the integrated output of accelerometers. The coordinate system is shown in
Figure 12 where x y z (down range, left, and up) is a launch fixed inertial
coordinate system. If the z axis is rotated counter-clockwise by an angle λ ,
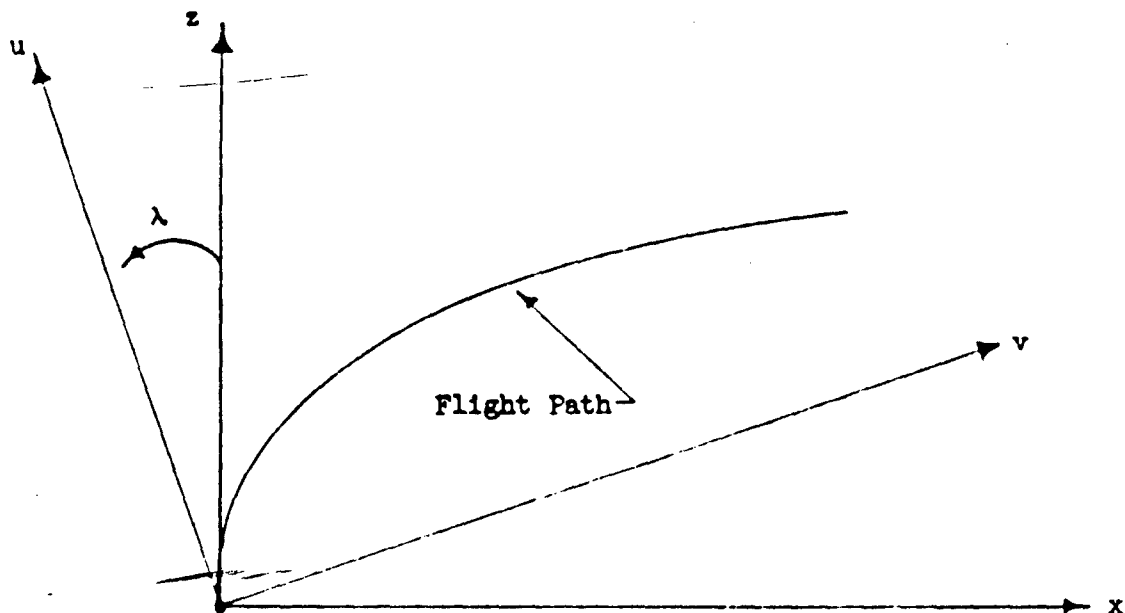

Figure 12 - COORDINATE SYSTEM FOR THRUST VELOCITY STEERING

the u axis is obtained. The integrated output of an accelerometer mounted
along u will differ from $\dot{u}$ at any instant of time by the initial value
of $\dot{u}$ and the integrated gravity term. If λ is equal to 90° minus the
constant attitude angle, it can be shown that for typical open loop trajectories
the actual integrated accelerometer output[*], $\dot{u}_a$, will have a time history
of the form shown in Figure 13.

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[*]Actually, of course, $\dot{u}_a$ may be the outputs of two or more accelerometers
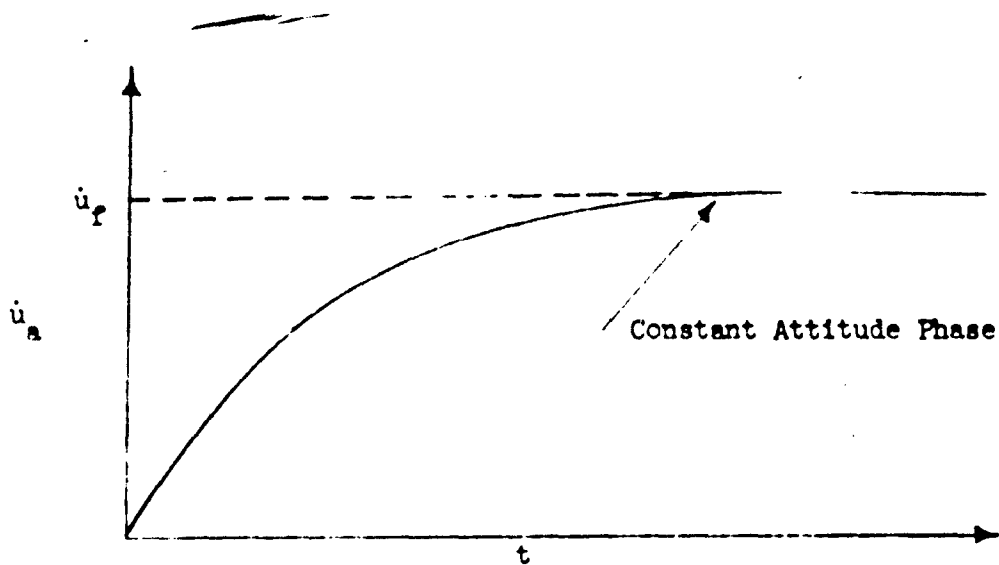suitably combined.

Figure 13 - INTEGRATED ACCELEROMETER OUTPUT FOR TYPICAL TRAJECTORY

The shape of the curve suggests an exponential approximation of the form

$$\dot{u}_a = \dot{u}_f (1 - e^{-\frac{t}{\tau_p}}) \tag{86}$$

If the vehicle can be caused to fly such that (86) is realized, then a gravity turn will result. This can be accomplished by commanding a pitch rate

$$\dot{\theta}_c = -K_p [\tau_p \ddot{u}_a + \dot{u}_a - \dot{u}_f] \tag{87}$$

If $\dot{\theta}_c/K_p$ is small, then (86) is an exact solution of (87). If it is desirable to command attitude rather than rate then (87) can be integrated and the command becomes

$$\theta_c = -K_p [\tau_p \dot{u}_a + u_a - u_f t + C] \tag{88}$$

It may be necessary to fit the $\dot{u}_a(t)$ vs t curve with two or more exponentials and to change constants at some convenient point such as a staging point. This is especially true if this type of steering is to be used over the entire flight rather than for the vacuum phase only. Velocity steering in yaw is accomplished by an attitude command of the form

$$\psi_c = -K_y[\tau_y \dot{y}_a + y_a] \tag{89}$$

Here there is no forcing function and the yaw steering loop is actually a nulling loop.

A somewhat more general pitch velocity steering scheme consists of controlling the flight path of the vehicle so that

$$V_z = f_1(V_x, x, z, t) \tag{90}$$

Again a variety of coordinate systems with and without gravity may be employed. An earth fixed coordinate system works particularly well, since the earth's atmosphere rotates with the earth and therefore is at rest with respect to the coordinate system. The steering necessary to realize (90) can be accomplished by attitude perturbations given by

$$\delta\theta_c = -K_p[V_z - f_1(V_x, x, z, t)] \tag{91}$$

where $\delta\theta/K_p$ is a small number. At the same time the nominal pitch program can be written as a function of one or more variables, e.g.

$$\theta_{nc} = f_2(V_x, x, z, t) \tag{92}$$

Combining (91) and (92) to get the total pitch attitude angle command gives

$$\theta_c = \theta_{nc} + \delta\theta_c = f_3(V_x, V_z, x, z, t) \tag{93}$$

In practice it is frequently found that the x , z , and t terms are not very effective and an expansion in V_x and V_z may be all that is necessary, e.g.

$$\theta_c = K_z V_z + K_x V_x + K_{xx} V_x^2 \tag{94}$$

The coefficients are obtained by selecting a number of points from the standard trajectory and using least square fitting procedures as was done for delta guidance constants. The difference here is that it is not necessary to use points from perturbed trajectories.

Velocity steering as typified by (88) and (94), although more complicated, has at least two advantages when compared with attitude-time programming. First the missile angle of attack due to a wind shear is reduced somewhat. This is due to the fact that the dynamic action of the loop causes the vehicle to "weathercock" into the wind. The second and primary advantage is that the trajectory perturbations due to vehicle perturbations are greatly reduced. When any type of perturbation guidance equations, e.g. delta, are used, this fact results in a reduction in the complexity of the equations and hence a simpler computer mechanization. On the debit side is the fact that this type of steering tends to "work" the attitude control system harder and as a result stability margins at vibration frequencies may be slightly reduced.

4.2 The Vacuum Phase

Once the vehicle is out of the atmosphere, structural constraints can be relaxed and the steering system can begin to perform its primary function, that of reducing $\bar{V}_g$ to zero. A steering or computational coordinate system is usually selected which aligns the x axis more or less with the desired thrust direction during the latter portion of powered flight.* The signal V_{gx} then represents the principal component of velocity-to-be-gained and V_{gy} and V_{gz} are considerably smaller.

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*The coordinate system used in Section 3 had the z axis aligned with launch vertical for purposes of simplicity. If pitch steering is done, it is usually necessary to tip the coordinate system.

One method which will cause the y and z components of $\bar{V}_g$ to vanish at cutoff is to drive these quantities continuously toward zero from the "initiation of guidance"[*]. This can be accomplished by interpreting V_{gy} and V_{gz} as error signals which are to be nulled by suitable attitude or attitude rate commands to the autopilot or attitude control system. Stability considerations usually require that some type or rate information also be included for damping purposes. A typical pitch attitude command would be of the form

$$\theta_c = \theta_o + K_1 V_{gz} + K_2 \dot{V}_{gz} \tag{95}$$

where θ_o is the nominal pitch angle. If a rate attitude command is to be used it would be of the form

$$\omega_p = K_3 V_{gz} + K_4 \dot{V}_{gz} \tag{96}$$

The yaw commands would be the same only, of course, V_{gz} would be replaced by V_{gy}. In order to reduce certain steady state errors it is often desirable to include also an integral term in the steering expressions.

An Important Modification - While a scheme as outlined above is capable of driving $\ddot{V}_g$ to zero at cutoff, it has one basic shortcoming. Even though the y and z components of $\bar{V}_g$ are small compared to V_{gx} at guidance initiation, they are still sufficiently large to cause excessive pitch and yaw commands and hence substantial maneuvering of the vehicle. As a result even a nominal vehicle is caused to deviate considerably from the reference trajectory and this is objectionable for two reasons: (1) It results in a non-optimum use of propellants and (2) It means that the required velocity computation must be accurate over a larger region.

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[*]"Initiation of guidance" is a term often used to indicate the time at which the vehicle steering is switched from the atmospheric phase to the vacuum phase.

The reason for the occurance of the above maneuvering becomes clear when it is realized that under reference conditions the y and z components of $\bar{V}_g$ differ substantially from zero prior to cutoff. Figure 14 gives the general shape of the three components for a typical long range rocket vehicle. The actual shapes will
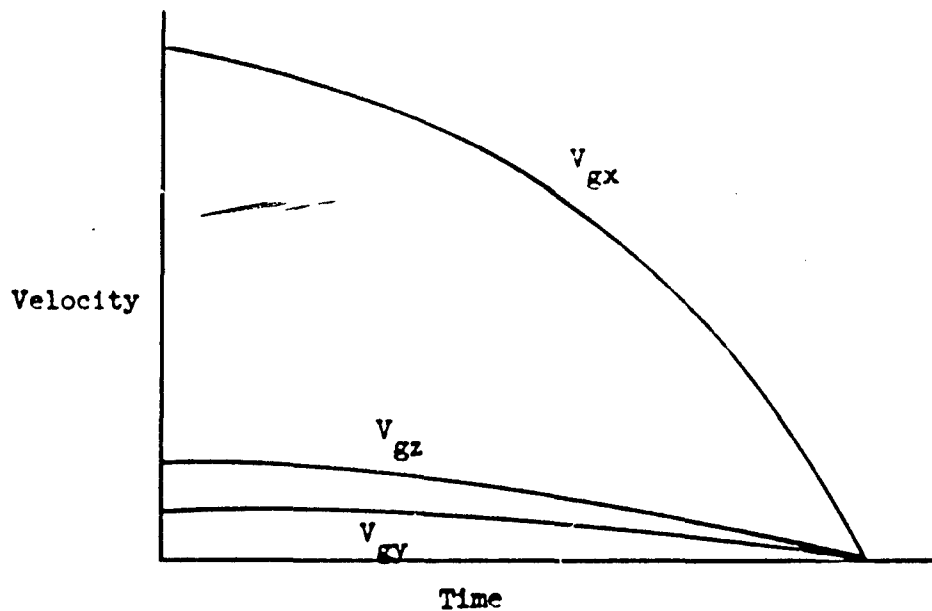

Figure 14 - GENERAL SHAPE OF $\bar{V}_g$ COMPONENTS

depend on the trajectory, coordinate system, and guidance constraints. The net result of this finding is that not V_{gy} and V_{gz} , but rather their deviations from nominal should be used as steering error signals. If this is done non-nominal missiles still "wander" to some extent, but not excessively so.

If the standard values of V_{gy} and V_{gz} are to be programmed, the question arises as to the independent variable against which they should be programmed. Time or velocity might be used, but further consideration leads to V_{gx} as the best choice. The pitch and yaw error signals are then of the form

$$V_{gz}^{\circledast} = V_{gz} - f(V_{gx}) \tag{97}$$

$$V_{gy}^{\circledast} = V_{gy} - f(V_{gx}) \tag{98}$$

where the function may be simply V_{gx} multiplied by a constant, but usually includes higher order terms. The function will be equal to zero at cutoff; hence the error signals become equal to V_{gz} and V_{gy} at cutoff. This, of course, is the dominant reason for using V_{gx} as the programming variable.

Cross Product Steering[*] - The steering method described in the preceeding paragraphs depends on the existance of a standard or reference trajectory. Frequently, especially when explicit guidance equations are used, the reference trajectory concept loses its meaning. It is then that more general steering methods must be employed. The best known of these is cross product steering where the vehicle is caused to have an attitude rate proportional to $\bar{V}_g \times \dot{\bar{V}}_g$.

The fundamental justification for the method begins with the definition of $\bar{V}_g$

$$\bar{V}_g = \bar{V}_R - \bar{V} \tag{99}$$

which when differentiated with respect to time gives

$$\dot{\bar{V}}_g = \dot{\bar{V}}_R - \dot{\bar{V}} \tag{100}$$

Since $\dot{\bar{V}}$ is missile acceleration which during powered flight is due to thrust and gravitational accelerations, (100) can be rewritten

$$\dot{\bar{V}}_g + a_T \bar{\xi} + (\bar{g} - \dot{\bar{V}}_R) = 0 \tag{101}$$

where $\bar{\xi}$ is the unit vector in the direction of $\bar{a}_T$, the thrust acceleration. Let $\bar{u}$ be a unit vector along $\bar{V}_g$. Therefore

$$\bar{V}_g = V_g \bar{u} \tag{102}$$

and

$$\dot{\bar{V}}_g = V_g \dot{\bar{u}} + \dot{V}_g \bar{u} \tag{103}$$

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[*]This Section is based mainly on work by J. M. Bachar, F. Baskin, and D. W. Whitcombe of Space Technology Laboratories.

Combining (101) with (103) gives

$$(V_g \dot{\bar{u}} + \dot{V}_g \bar{u}) + a_T \bar{\xi} + (\bar{g} - \dot{\bar{V}}_R) = 0 \tag{104}$$

Taking the dot product of (104) and $\bar{u}$ results in

$$\dot{V}_g + a_T(\bar{\xi} \cdot \bar{u}) + [(\bar{g} - \dot{\bar{V}}_R) \cdot \bar{u}] = 0 \tag{105}$$

since $\bar{u} \cdot \dot{\bar{u}} = 0$ and $\bar{u} \cdot \bar{u} = 1$. If then the thrust is directed along the $\bar{V}_g$ direction, i.e., $\bar{\xi} \triangleq \bar{u}$, (105) is reduced to the scalar equation

$$\dot{V}_g + a_T + [(\bar{g} - \dot{\bar{V}}_R) \cdot \bar{u}] = 0 \tag{106}$$

The necessary and sufficient condition for the monotonic decrease of $\bar{V}_g$ to zero is that $\dot{\bar{V}}_g$ always be negative. This indeed is the case as long as

$$[(\bar{g} - \dot{\bar{V}}_R) \cdot \bar{u}] < a_T \tag{107}$$

which in turn is true for current high acceleration rocket vehicles. Hence it has been shown that causing the thrust vector to be pointed along the $\bar{V}_g$ vector guarantees that $\bar{V}_g$ will be driven to zero, i.e., all three components.

What then does this attitude requirement imply with regard to vehicle attitude rates? As was stated in (102)

$$\bar{u} = \frac{\bar{V}_g}{V_g} \tag{108}$$

The time derivative of (108) is shown to be

$$\dot{\bar{u}} = \frac{V_g \dot{\bar{V}}_g - \dot{V}_g \bar{V}_g}{V_g^2} \tag{109}$$

Since $\bar{u}$ is a unit vector, it is true that

$$\bar{\omega} = \bar{u} \times \dot{\bar{u}} \tag{110}$$

and hence

$$\bar{\omega} = \frac{\bar{V}_g}{V_g} \times \frac{V_g \dot{\bar{V}}_g}{V_g^2} - \frac{\bar{V}_g}{V_g} \times \frac{\dot{V}_g \bar{V}_g}{V_g^2} \tag{111}$$

The second term is identically zero so the turning rate of the $\bar{V}_g$ vector is equal to

$$\bar{\omega} = \frac{\bar{V}_g \times \dot{\bar{V}}_g}{V_g^2} \tag{112}$$

If the thrust vector is to be kept aligned with the $\bar{V}_g$ vector $(\bar{\xi} = \bar{u})$, then the missile turning rate must be equal to (112).

An examination of (112) reveals that the $\bar{V}_g$ vector's turning rate becomes infinite at the instant of cutoff. Since no physical missile has this capability, there will always be some steering error at cutoff with this approach. This situation appears to be analogous to a pure pursuit course where infinite turning rates are also encountered at impact. In that case the situation is alliviated by modifying the course to include a lead angle. In the case of steering a similar effect is achieved by biasing_out the nominal error, since the error turns out to be relatively independent of trajectory perturbations.

In practice the commanded vehicle turning rate is of the form

$$\omega_c = K[\bar{V}_g \times \dot{\bar{V}}_g] \tag{113}$$

where K may or may not be a function of V_g. The pitch and yaw commanded rates will be equal to

$$\omega_p = -K[V_{gx}\dot{V}_{gz} - V_{gz}\dot{V}_{gx}] \tag{114}$$

$$\omega_y = K[V_{gx}\dot{V}_{gy} - V_{gy}\dot{V}_{gx}] \tag{115}$$

The corresponding roll rate contains redundant information and is not used. An equivalent system which commands angles rather than rates is possible, but will not be described.

The cross product steering scheme causes the vehicle to fly a nearly constant attitude trajectory which is desirable from a fuel standpoint. At guidance initiations, the system commands rather large rates until the proper attitude is obtained and makes only small changes thereafter. The system does not attempt to remove y or z components early, rather it attempts to drive all components to zero simultaneously.

4.3 Thrust Termination

It has been shown that it is possible to steer a rocket vehicle in a way which guarantees that all three components of the velocity-to-be-gained will reach zero at the same instant, at which time thrust is terminated. If the steering system is perfect, i.e., all three components of $\bar{V}_g$ become uniquely zero, then the magnitude of $\bar{V}_g$ or the magnitude of any component of $\bar{V}_g$ is useful as a cutoff signal. In a practical system where steering errors will occur due to both static and dynamic effects, the signal is usually given when

$$V_{gx} < K_{co} \tag{116}$$

where K_{co} is some allowable tolerance on cutoff velocity. The V_{gx} component is chosen for a number of reasons:

(1) The coordinate system is usually chosen so that $|V_{gx}|$ is nearly equal to $|V_g|$.

(2) V_{gx} is usually already available since it is necessary for steering purposes.

(3) If either V_{gy} or V_{gz} were used, small errors in these quantities could cause large cutoff errors, since usually $\dot{V}_{gx} \gg \dot{V}_{gy}$ or $\dot{V}_{gz}$.

Further consideration of the cutoff problem depends on the type of engine employed.

System With Liquid Engines - Thrust is terminated in a liquid propellant engine by closing valves in the fuel and oxidizer lines. Since the lines past the valves and the thrust chamber itself contain a certain amount of the liquids, thrust actually continues for a fraction of a second after valve closure, a typical value being 0.2 seconds. If the acceleration near burnout is 10 g's, and a linear decay is assumed, 32 fps is added after valve closure corresponding to 32 miles miss in the case of a 5500 mile missile.

A known residual impulse can, of course, be handled by anticipating its effect and calling for thrust termination when V_{gx} is still positive by the required amount. The real difficulty comes from the fact that there is some uncertainty attached to the residual impulse due to valve operating times and other factors. A typical number here might be 15% of the total residual impulse, or 4.8 fps. Since velocity uncertainties of this size are not tolerable in a high quality guidance system, liquid engines are frequently equipped with vernier engines.

Vernier engines are small liquid or solid engines which are capable of accelerating the vehicle at 0.1 - 0.2 g's. When a vehicle is equipped with verniers, the main engine is shut down with V_{gx} positive by an amount which is larger than any ΔV_{gx} which is expected from a residual impulse. The vernier is then ignited (if a liquid it has probably been on with the main engine) and its thrust is terminated when V_{gx} is positive by an amount which compensates for its anticipated residual impulse. While the shut down procedure is the same for the small engine, the difference lies in the fact that the velocity uncertainty is reduced by the ratio of the thrusts of the main and vernier engines. For the 0.1 g vernier, this means that the velocity, for the number chosen, would be reduced by a factor of 100 to 0.048 fps, which would usually be called negligible. For medium accuracy systems, the added complexity of the vernier system often precludes its use.

Cutoff Extrapolation - Even when a vernier system is used, there still remains the problem of generating the thrust termination signal with sufficiently small time quantization. For many inertial guidance computers, position, velocity, and velocity-to-be-gained are computed once every one half second. In the case of a 0.1 g vernier, the one quarter second uncertainty could cause a 0.8 fps velocity error. Shutting down the main engine with a time uncertainty this large would also require the use of a vernier with longer mean burning time. In order to reduce these effects, an extrapolation procedure is used. Consider for example a system where V_{gx} is generated every one half second and where the extrapolation is accomplished every one sixteenth of a second. Then

$$V_{gx(N+n)} = V_{gxN} + \frac{n}{16} \dot{V}_{gx} \tag{117}$$

where V_{gxN} is the value of V_{gx} at a half second point and $V_{gx(N+n)}$ is the value $n/16$ seconds later. Sufficient accuracy is usually obtained by using the standard value of $\dot{V}_{gx}$ so the main engine is cutoff when

$$V_{gxN} + nK_1 \leq K_{meco} \tag{118}$$

and the vernier engine is cutoff when

$$V_{gxN} + nK_2 \leq K_{veco} \tag{119}$$

where K_1 and K_2 are standard values of $\dot{V}_{gxo}/16$ before main engine and vernier cutoff and K_{meco} and K_{veco} are computed to compensate for nominal values of residual impulse plus an uncertainty in the case of the main engine. In practice, the nominal vernier period should be long enough to damp out any steering transients. Usually the required length is in the 20-30 second region.

Systems With Solid Engines - Solid propellant engines differ from liquid propellant engines in that combustion occurs in the full length of the engine case, rather than in the thrust chamber only. Thrust is terminated in a solid engine by blowing holes in the forward end of the case. This has the effect of reducing the thrust to zero in a very short period of time. Even with large engines, the residual impulse uncertainty can be reduced to the point where verniers are not required.

Cutting off a high acceleration engine does however impose some rather stringent requirements on the cutoff time granularity. For instance for a 10 g engine, cutting off to $\pm$ 1/32 second would result in a velocity uncertainty of 10 fps. If the time granularity is decreased to a half millisecond, then the velocity uncertainty is decreased to 0.16 fps. Therefore some form of extrapolation is required, since the basic guidance computations are done at a much slower rate. Here it turns out that good results are obtained when the engine is cutoff when

$$ V_{gx} = V_{gxo} + \int_{t_o}^{t} a_{Tx} \, dx \leq K_{meco} \tag{120} $$

where V_{gxo} is the value of V_{gx} at time t_o where the extrapolation mode is begun and where K_{meco} is a constant which compensates for a nominal residual impulse. The extrapolation assumes that V_{Rx} and the integral of gravitational acceleration change a negligible amount over the extrapolation interval which is usually true for intervals of the order of a half second.

Multiple Thrust Periods - As has been pointed out previously, more complicated missions such as placing a satellite into a high altitude orbit often have two or more thrust periods. During the various thrust period, the nominal thrust direction in general has different orientations with respect to V_{gx}, since the computational coordinate system usually remains fixed. In this case it is necessary to resolve either $\bar{V}_g$ or the steering commands themselves from inertial to vehicle coordinates. This can be done either in the computer or by means of an analog resolver chain. Cutoff is generated on the basis of some major component of $\bar{V}_g$.